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NUMERICAL METHODS

Construction and Structure Properties of Solutions of a Periodic Boundary Value Problem for a Generalization of the Matrix Lyapunov and Riccati Equations

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Abstract—We obtain constructive sufficient conditions for the unique solvability of a periodic boundary value problem for a matrix differential equation that generalizes the Lyapunov and Riccati equations, develop an algorithm for constructing the solution of this equation, estimate the domain where the solution is localized, and study the structural properties of the solution.

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We study the boundary value problem

$$\frac{dX}{dt} = A(t)X + XB(t) + XQ(t)X + F(t,X), \qquad X = X(t) \in \mathbb{R}^{n\times n}, \tag{1}$$

$$X(0) = X(\omega); \tag{2}$$

here $t \in I$, $A, B, Q \in C(I, \mathbb{R}^{n\times n})$, and $F \in C(D_{\tilde\rho}, \mathbb{R}^{n\times n})$, where $D_{\tilde\rho} = \{(t,X) : t \in I,\ \|X\| < \tilde\rho\}$, $I = [0,\omega]$, $\omega > 0$, $0 < \tilde\rho \le \infty$, and $\|\cdot\|$ is a given matrix norm. We assume that the matrix function $F(t,X)$ (locally) satisfies the Lipschitz condition in the variable X in the domain $D_{\tilde\rho}$ and $F(t,0) \not\equiv 0$.

For $Q = 0$, the two-point boundary value problem was studied in [1] by qualitative methods, in [3, 4] by constructive methods [2], and in [5–7] for the case of periodic boundary conditions. The periodic boundary value problem for the Riccati equation was considered in the monograph [2, p. 277], and the two-point problem was considered in [8]. In the present paper, we mainly use the constructive method in [9]. Our results generalize and develop the corresponding results presented in [5–7], but our conditions for unique solvability are constructive, the algorithms for constructing the solutions are sufficiently convenient in applications, and the method used here differs from the methods [2, Ch. 3–4] applied in [3–8]. The cited papers use two-sided regularization related to the construction of a solution of a matrix algebraic Lyapunov equation, the solution methods being given in [10–14]. The structure properties of matrix differential equations of various types and their solutions were studied in [15–17].

In the present paper, we study the unique solvability of problem (1), (2), the construction of the solution, and the structure properties of the exact and approximate solutions. This problem was studied in [18] by the method given in the monograph [2, Ch. 3] for the case mainly related to the properties of the matrix $B(t)$.

Just as in [9], we seek the solution of problem (1), (2) in the form

$$X(t) = C + Y(t), \tag{3}$$

where C is a constant matrix and $Y(t)$, $t \in I$, is a matrix subjected to the conditions

$$Y(0) = Y(\omega), \tag{4}$$

$$\int_0^\omega [A(\tau)Y(\tau) + Y(\tau)B(\tau)]\, d\tau = 0. \tag{5}$$

We introduce the following notation:

$$M = \int_0^\omega A(\tau)\, d\tau, \quad N = -\int_0^\omega B(\tau)\, d\tau, \quad \gamma = \|\Phi^{-1}\|, \quad \alpha = \max_t \|A(t)\|, \quad \beta = \max_t \|B(t)\|,$$

$$\delta = \max_t \|Q(t)\|, \qquad h = \max_t \|F(t,0)\|, \qquad \|X\|_C = \max_t \|X(t)\|,$$

$$D_\rho = \{(t,X) : 0 \le t \le \omega, \ \ \|X\| \le \rho\}, \qquad D = \{X(t) : \|X\|_C \le \rho\},$$

$$\varphi(\rho) = \gamma\delta\omega[1 + 0.5(\alpha+\beta)\omega]\rho^2 + \gamma\omega[L + 0.5(\alpha+\beta)(\alpha+\beta+L)\omega]\rho + \gamma\omega h[1 + 0.5(\alpha+\beta)\omega],$$

$$\varphi_1(\rho) = \gamma\omega(\delta\rho^2 + L\rho + h), \qquad \varphi_2(\rho) = 0.5\gamma(\alpha+\beta)\omega^2[\delta\rho^2 + (\alpha+\beta+L)\rho + h],$$

$$q(\rho) = \gamma\delta\omega[(\alpha+\beta)\omega + 2]\rho + 0.5\gamma(\alpha+\beta)(\alpha+\beta+L)\omega^2 + \gamma L\omega,$$

$$q_1(\rho) = \gamma\omega(2\delta\rho + L), \qquad q_2(\rho) = 0.5\gamma(\alpha+\beta)(\alpha+\beta+2\delta\rho+L)\omega^2,$$

where $0 < \rho < \tilde{\rho}$, $t \in I$, $L = L(\rho) > 0$ is the Lipschitz constant of the matrix function $F(t,X)$ in the domain D_ρ, the linear operator Φ is defined by the formula $\Phi Z = MZ - ZN$, and $\|\cdot\|$ is a matrix norm coordinated in the sense of [10, p. 410], for example, any of the norms given in the monograph [19, p. 21].

Theorem 1. *Assume that the matrices M and N do not have common characteristic numbers and the following conditions are satisfied:*

$$\varphi(\rho) \le \rho, \tag{6}$$

$$q(\rho) < 1. \tag{7}$$

Then there exists a unique solution of problem (1), (2) *in the domain D_ρ. This solution can be represented in the form* (3), *and one has the estimates $\|C\| \le \varphi_1(\rho)$ and $\|Y\|_C \le \varphi_2(\rho)$.*

Proof. Let us obtain a system of matrix equations equivalent to problem (1), (2) under assumptions (3)–(5). Suppose that this problem is solvable and has a solution (3) satisfying conditions (4) and (5).

It follows from Eq. (1) in view of assumption (3) and condition (4) that

$$\int_0^\omega [A(\tau)(C + Y(\tau)) + (C + Y(\tau))B(\tau) + (C + Y(\tau))Q(\tau)(C + Y(\tau)) + F(\tau, C + Y(\tau))]\, d\tau = 0. \tag{8}$$

We use condition (5) to write Eq. (8) in the form

$$MC - CN = -\int_0^\omega [(C + Y(\tau))Q(\tau)(C + Y(\tau)) + F(\tau, C + Y(\tau))]\, d\tau = 0.$$

Since the matrices M and N do not have common characteristic numbers, it follows from [10, p. 207] that the operator Φ is uniquely invertible and hence the inverse operator Φ^{-1} (bounded and linear) is well defined. Based on the invertibility of the operator Φ, we obtain the matrix equation

$$C = -\Phi^{-1} \int_0^\omega [(C + Y(\tau))Q(\tau)(C + Y(\tau)) + F(\tau, C + Y(\tau))]\, d\tau. \tag{9}$$

Further, we use the identities [2, p. 47]

$$\int_0^\omega A(\tau)Y(\tau)\,d\tau = MY(t) - \int_0^t \left(\int_0^\tau A(\sigma)\,d\sigma\right) dY(\tau) + \int_t^\omega \left(\int_\tau^\omega A(\sigma)\,d\sigma\right) dY(\tau), \tag{10}$$

$$\int_0^\omega Y(\tau)B(\tau)\,d\tau = -Y(t)N - \int_0^t dY(\tau)\left(\int_0^\tau B(\sigma)\,d\sigma\right) + \int_t^\omega dY(\tau)\left(\int_\tau^\omega B(\sigma)\,d\sigma\right) \tag{11}$$

in the integral condition (5). Based on identities (10) and (11), we see that, by Eq. (1), condition (5) implies the matrix integral equation

$$\begin{aligned} MY(t) - Y(t)N &= \int_0^t \left(\int_0^\tau A(\sigma)\,d\sigma\right) S(\tau)\,d\tau - \int_t^\omega \left(\int_\tau^\omega A(\sigma)\,d\sigma\right) S(\tau)\,d\tau \\ &\quad + \int_0^t S(\tau)\left(\int_0^\tau B(\sigma)\,d\sigma\right) d\tau - \int_t^\omega S(\tau)\left(\int_\tau^\omega B(\sigma)\,d\sigma\right) d\tau, \end{aligned} \tag{12}$$

where

$$S(\tau) = A(\tau)(C+Y(\tau)) + (C+Y(\tau))B(\tau) + (C+Y(\tau))Q(\tau)(C+Y(\tau)) + F(\tau, C+Y(\tau)).$$

We use the invertibility of the operator Φ to write Eq. (12) in the form

$$Y(t) = \Phi^{-1}\int_0^\omega [K_A(t,\tau)S(\tau) + S(\tau)K_B(t,\tau)]\,d\tau, \tag{13}$$

where

$$K_H(t,\tau) = \begin{cases} \displaystyle\int_0^\tau H(\sigma)\,d\sigma, & 0 \le \tau \le t \le \omega, \\ \displaystyle -\int_\tau^\omega H(\sigma)\,d\sigma, & 0 \le t < \tau \le \omega, \end{cases} \qquad H = A, B.$$

Thus, C and $Y(t)$ satisfy relations (9) and (13).

The converse is true as well: each continuous solution of Eqs. (9), (13) is a solution of problem (1), (2) in the form (3) with conditions (4), (5). This can be proved by simple calculations.

Let us study the solvability of this system of equations. In operator form, system (9), (13) becomes

$$C = \mathcal{L}_1(C, Y), \tag{14}$$

$$Y = \mathcal{L}_2(C, Y), \tag{15}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the corresponding integral operators in Eqs. (9) and (13). The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ act from the space $\mathbb{R}^{n\times n} \times C(I, \mathbb{R}^{n\times n})$ to the spaces $\mathbb{R}^{n\times n}$ and $C(I, \mathbb{R}^{n\times n})$, respectively.

Let us show that it follows from conditions (6), (7) that a modification of the Banach–Caccioppoli contraction mapping principle is satisfied on the set

$$\tilde{D} = \{(C, Y(t)) : \|C\| \le \varphi_1(\rho),\ \|Y\|_C \le \varphi_2(\rho)\}.$$

Note that $\varphi_1(\rho) + \varphi_2(\rho) = \varphi(\rho)$.

First, we prove that $(\mathcal{L}_1(C,Y),\mathcal{L}_2(C,Y))\in\tilde{D}$ if $(C,Y(t))\in\tilde{D}$. Passing to the norms in Eq. (14), we obtain the chain of inequalities

$$\begin{aligned}\|\mathcal{L}_1(C,Y)\| &\le \|\Phi^{-1}\|\int_0^\omega[\|C+Y(\tau)\|\|Q(\tau)\|\|C+Y(\tau)\|+\|F(\tau,C+Y(\tau))\|]\,d\tau\\ &\le \|\Phi^{-1}\|\int_0^\omega[(\|C\|+\|Y(\tau)\|)^2\|Q(\tau)\|+L(\|C\|+\|Y(\tau)\|)+\|F(\tau,0)\|]\,d\tau\\ &\le \gamma\omega[\delta\varphi^2(\rho)+L\varphi(\rho)+h]\le\gamma\omega(\delta\rho^2+L\rho+h)=\varphi_1(\rho).\end{aligned}$$

Thus, we have the estimate

$$\|\mathcal{L}_1(C,Y)\|\le\varphi_1(\rho). \tag{16}$$

Further, we perform similar estimates in Eq. (15),

$$\begin{aligned}\|\mathcal{L}_2(C,Y)\| &\le \|\Phi^{-1}\|\left\|\int_0^\omega[K_A(t,\tau)S(\tau)+S(\tau)K_B(t,\tau)]\,d\tau\right\|\\ &\le \gamma\int_0^\omega[\|K_A(t,\tau)\|+\|K_B(t,\tau)\|][(\|A(\tau)\|+\|B(\tau)\|)(\|C\|+\|Y(\tau)\|)\\ &\quad+(\|C\|+\|Y(\tau)\|)^2\|Q(\tau)\|+L(\|C\|+\|Y(\tau)\|)+\|F(\tau,0)\|]\,d\tau\\ &\le \gamma[\delta(\varphi_1(\rho)+\varphi_2(\rho))^2+(\alpha+\beta+L)(\varphi_1(\rho)+\varphi_2(\rho))+h]\\ &\quad\times\int_0^\omega[\|K_A(t,\tau)\|+\|K_B(t,\tau)\|]\,d\tau\\ &= \gamma[\delta\varphi^2(\rho)+(\alpha+\beta+L)\varphi(\rho)+h]\int_0^\omega[\|K_A(t,\tau)\|+\|K_B(t,\tau)\|]\,d\tau\\ &\le 0.5\gamma(\alpha+\beta)[\delta\rho^2+(\alpha+\beta+L)\rho+h]\omega^2=\varphi_2(\rho).\end{aligned}$$

This implies the estimate

$$\|\mathcal{L}_2(C,Y)\|_C\le\varphi_2(\rho). \tag{17}$$

Further, from Eq. (14) for arbitrary $(\tilde{C},\tilde{Y}),(\bar{C},\bar{Y})\in\tilde{D}$ we obtain the chain of inequalities

$$\begin{aligned}\|\mathcal{L}_1(\bar{C},\bar{Y})-\mathcal{L}_1(\tilde{C},\tilde{Y})\| &\le \|\Phi^{-1}\|\int_0^\omega[\|(\bar{C}+\bar{Y}(\tau))Q(\tau)(\bar{C}+\bar{Y}(\tau))-(\tilde{C}+\tilde{Y}(\tau))Q(\tau)(\tilde{C}+\tilde{Y}(\tau))\|\\ &\quad+\|F(\tau,\bar{C}+\bar{Y}(\tau)-F(\tau,\tilde{C}+\tilde{Y}(\tau))\|]\,d\tau\\ &\le \gamma\int_0^\omega[2\delta\rho(\|\bar{C}-\tilde{C}\|+\|\bar{Y}(\tau)-\tilde{Y}(\tau)\|)+L(\|\bar{C}-\tilde{C}\|+\|\bar{Y}(\tau)-\tilde{Y}(\tau)\|)]\,d\tau\\ &\le q_1(\rho)(\|\bar{C}-\tilde{C}\|+\|\bar{Y}-\tilde{Y}\|_C).\end{aligned}$$

Therefore, we have the estimate

$$\|\mathcal{L}_1(\bar{C},\bar{Y})-\mathcal{L}_1(\tilde{C},\tilde{Y})\|\le q_1(\rho)(\|\bar{C}-\tilde{C}\|+\|\bar{Y}-\tilde{Y}\|_C). \tag{18}$$

In a similar way, we obtain the following estimate based on (16) :

$$\|\mathcal{L}_2(\bar{C},\bar{Y}) - \mathcal{L}_2(\tilde{C},\tilde{Y})\| \le \|\Phi^{-1}\| \int_0^\omega \|K_A(t,\tau)(\bar{S}(\tau) - \tilde{S}(\tau)) + (\bar{S}(\tau) - \tilde{S}(\tau))K_B(t,\tau)\|\, d\tau$$

$$\le \|\Phi^{-1}\| \int_0^\omega [\|K_A(t,\tau)\| + \|K_B(t,\tau)\|]\|\bar{S}(\tau) - \tilde{S}(\tau)\|\, d\tau$$

$$\le \gamma(\alpha+\beta+2\delta\rho+L) \int_0^\omega (\|K_A(t,\tau)\| + \|K_B(t,\tau)\|)(\|\bar{C}-\tilde{C}\| + \|\bar{Y}(\tau) - \tilde{Y}(\tau)\|)\, d\tau$$

$$\le 0.5\gamma(\alpha+\beta)(\alpha+\beta+2\delta\rho+L)\omega^2(\|\bar{C}-\tilde{C}\| + \|\bar{Y}-\tilde{Y}\|_C)$$

$$= q_2(\rho)(\|\bar{C}-\tilde{C}\| + \|\bar{Y}-\tilde{Y}\|_C),$$

where

$$\tilde{S}(\tau) = A(\tau)(\tilde{C}+\tilde{Y}(\tau)) + (\tilde{C}+\tilde{Y}(\tau))B(\tau) + (\tilde{C}+\tilde{Y}(\tau))Q(\tau)(\tilde{C}+\tilde{Y}(\tau)) + F(\tau,\tilde{C}+\tilde{Y}(\tau)).$$

This implies the estimate

$$\|\mathcal{L}_2(\bar{C},\bar{Y}) - \mathcal{L}_2(\tilde{C},\tilde{Y})\|_C \le q_2(\rho)[\|\bar{C}-\tilde{C}\| + \|\bar{Y}(\tau) - \tilde{Y}(\tau)\|_C]. \tag{19}$$

Note that $q_1(\rho) + q_2(\rho) = q(\rho)$. Let us write the estimates (18), (19) in the matrix form

$$\tilde{H} \le QH, \tag{20}$$

where

$$\tilde{H} = \begin{pmatrix} \|\mathcal{L}_1(\bar{C},\bar{Y}) - \mathcal{L}_1(\tilde{C},\tilde{Y})\| \\ \|\mathcal{L}_2(\bar{C},\bar{Y}) - \mathcal{L}_2(\tilde{C},\tilde{Y})\|_C \end{pmatrix}, \qquad H = \begin{pmatrix} \|\bar{C}-\tilde{C}\| \\ \|\bar{Y}(\tau) - \tilde{Y}(\tau)\|_C \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1(\rho) & q_1(\rho) \\ q_2(\rho) & q_2(\rho) \end{pmatrix}.$$

Condition (7) implies the relation $\det(E-Q) > 0$, where $E = \operatorname{diag}[1,1]$. Based on [10, p. 370], we conclude that the characteristic numbers of the positive matrix Q lie inside the unit circle centered at the origin and the matrix $E-Q$ is positively invertible. Thus, we see that relations (16)–(19), which are a modification [20, p. 94] of the Banach–Caccioppoli contraction mapping principle [21, p. 605], hold on the set $\tilde{D}$ for system (14), (15). It follows that there exists a unique solution of this system on the set $\tilde{D}$. This means that problem (1), (2) in the form (3) with conditions (4), (5) is uniquely solvable in the domain D_ρ. The proof of the theorem is complete.

Remark 1. In contrast to the method presented in the monograph [2, Ch. 4], our approach is based on the use [with the help of relation (5)] of appropriate functional properties of the matrices $A(t)$ and $B(t)$ on the right-hand side in Eq. (1).

If the matrices $A(t)$ and $B(t)$ are constant, then condition (5) becomes

$$A\int_0^\omega Y(\tau)\, d\tau + \int_0^\omega Y(\tau)\, d\tau B = 0,$$

whence, in view of the fact that the matrices A and B do not have common characteristic numbers by Theorem 1, we obtain

$$\int_0^\omega Y(\tau)\, d\tau = 0.$$

In this case, condition (5) corresponds to a Fourier type method [2, Ch. 4].

Remark 2. If we take the matrix norm (maximum column norm) [19, p. 21]

$$\|D\|_{II} = \max_m \sum_{k=1}^{n} |d_{km}|, \quad \text{where} \quad D = (d_{km})_{k,m=1}^{n},$$

for the norm in inequality (20), then this inequality implies the estimate $\|\tilde{H}\|_{II} \le q(\rho)\|H\|_{II}$, because $\|Q\|_{II} = q_1(\rho) + q_2(\rho) = q(\rho) < 1$.

To construct the solution, we use an algorithm with an explicit computational scheme. For problem (1), (2), this scheme is given by the formulas $X_k(t) = C_k + Y_k(t)$,

$$\begin{aligned}\frac{dY_{k+2}(t)}{dt} &= A(t)(C_{k+1} + Y_{k+1}(t)) + (C_{k+1} + Y_{k+1}(t))B(t)\\ &\quad + (C_k + Y_k(t))Q(t)(C_k + Y_k(t)) + F(t, C_k + Y_k(t)), \qquad (21)\\ Y_{k+2}(0) &= Y_{k+2}(\omega), \qquad (22)\\ \int_0^\omega &[A(\tau)Y_{k+1}(\tau) + Y_{k+1}(\tau)B(\tau)]\,d\tau = 0, \qquad k = 0, 1, 2, \ldots, \qquad (23)\end{aligned}$$

where we take zero matrices for the initial approximation C_0, Y_0 and the next approximation C_1, Y_1 is sought in the form of constant matrices that give approximate solutions satisfying conditions (22) and (23), respectively; here $Y_1 = 0$. The approximation C_1 is determined from the matrix algebraic equation

$$\int_0^\omega A(\tau)\,d\tau C_1 + C_1 \int_0^\omega B(\tau)\,d\tau + \int_0^\omega F(\tau, 0)\,d\tau = 0,$$

whose solution has the form

$$C_1 = -\Phi^{-1} \int_0^\omega F(\tau, 0)\,d\tau.$$

Based on relations (21)–(23), let us derive the corresponding equivalent recursion integral relations for the sequence of approximate solutions. To this end, we first obtain

$$C_{k+1} = -\Phi^{-1} \int_0^\omega [(C_k + Y_k(\tau))Q(\tau)(C_k + Y_k(\tau)) + F(\tau, C_k + Y_k(\tau))]\,d\tau, \qquad k = 1, 2, \ldots, \qquad (24)$$

from Eqs. (21) and (22) with regard to (23) and then find $Y_{k+1}(t)$; namely, we use identities (10) and (11) and condition (23) with regard to (21) to obtain

$$\begin{aligned}Y_{k+1}(t) = \Phi^{-1}\Big[&\int_0^t \Big(\int_0^\tau A(\sigma)\,d\sigma\Big) S_{k,k-1}(\tau)\,d\tau - \int_t^\omega \Big(\int_\tau^\omega A(\sigma)\,d\sigma\Big) S_{k,k-1}(\tau)\,d\tau\\ &+ \int_0^t S_{k,k-1}(\tau)\Big(\int_0^\tau B(\sigma)\,d\sigma\Big)\,d\tau - \int_t^\omega S_{k,k-1}(\tau)\Big(\int_\tau^\omega B(\sigma)\,d\sigma\Big)\,d\tau\Big],\end{aligned}$$

or

$$Y_{k+1}(t) = \Phi^{-1} \int_0^\omega [K_A(t, \tau)S_{k,k-1}(\tau) + S_{k,k-1}(\tau)K_B(t, \tau)]\,d\tau, \qquad k = 1, 2, \ldots, \qquad (25)$$

where

$$S_{k,k-1}(\tau) = A(\tau)(C_k + Y_k(\tau)) + (C_k + Y_k(\tau))B(\tau) + (C_{k-1} + Y_{k-1}(\tau))Q(\tau)(C_{k-1} + Y_{k-1}(\tau)) + F(\tau, C_{k-1} + Y_{k-1}(\tau)).$$

We see that the computational scheme of the algorithm (24), (25) is explicit and, to construct approximate solutions by this algorithm, one needs to carry out relatively simple computations. As a results, we construct the sequence $((C_k, Y_k(t)))_{k=0}^{\infty}$. Using condition (6) and proceeding by induction on k, we can readily prove that all terms of this sequence belong to the set $\tilde{D}$. When doing so, we must apply the corresponding estimates of the form (16), (17) to the algorithm (24), (25). To this end, we first pass to the norms in relation (24),

$$\|C_{k+1}\| \le \|\Phi^{-1}\| \int_0^{\omega} [\|C_k + Y_k(\tau)\|\|Q(\tau)\|\|C_k + Y_k(\tau)\| + \|F(\tau, C_k + Y_k(\tau))\|]\, d\tau \le \gamma\omega[\delta(\|C_k\| + \|Y_k\|_C)^2 + L(\|C_k\| + \|Y_k\|_C) + h], \qquad k = 0, 1, 2, \ldots \tag{26}$$

In a similar way, for (25) we have

$$\|Y_{k+1}(t)\| \le \|\Phi^{-1}\| \int_0^{\omega} [\|K_A(t,\tau)\| + \|K_B(t,\tau)\|]\|S_{k,k-1}(\tau)\|\, d\tau$$

$$\le \gamma \int_0^{\omega} [\|K_A(t,\tau)\| + \|K_B(t,\tau)\|][(\alpha+\beta)\|C_k + Y_k(\tau)\| + \delta\|C_{k-1} + Y_{k-1}(\tau)\|^2 + \|F(\tau, C_{k-1} + Y_{k-1}(\tau))\|]\, d\tau \le 0.5\gamma(\alpha+\beta)\omega^2[\delta(\|C_{k-1}\| + \|Y_{k-1}\|_C)^2 + (\alpha+\beta)(\|C_k\| + Y_k\|_C) + L(\|C_{k-1}\| + \|Y_{k-1}\|_C) + h], \qquad k = 1, 2, \ldots,$$

which implies the estimate

$$\|Y_{k+1}\|_C \le 0.5\gamma(\alpha+\beta)\omega^2[\delta(\|C_{k-1}\| + \|Y_{k-1}\|_C)^2 + (\alpha+\beta)(\|C_k\| + \|Y_k\|_C) + L(\|C_{k-1}\| + \|Y_{k-1}\|_C) + h], \qquad k = 1, 2, \ldots \tag{27}$$

Since $(C_0, Y_0(t)) \in \tilde{D}$ and $(C_1, Y_1(t)) \in \tilde{D}$, we obtain the desired inequalities $\|C_m\| \le \varphi_1(\rho)$, $\|Y_m\|_C \le \varphi_2(\rho)$ $(m = 0, 1, 2, \ldots)$ from the estimates (26), (27) with regard to condition (6) by induction.

Now let us study the convergence of the sequence $((C_k, Y_k(t)))_{k=0}^{\infty}$. Following the well-known method (e.g., see [22, p. 54]), we replace this problem by the equivalent problem on the convergence of the series

$$C_0 + (C_1 - C_0) + (C_2 - C_1) + \cdots + (C_k - C_{k-1}) + \cdots,$$
$$Y_0(t) + (Y_1(t) - Y_0(t)) + (Y_2(t) - Y_1(t)) + \cdots + (Y_k(t) - Y_{k-1}(t)) + \cdots$$

To this end, we construct a converging matrix series that majorizes these matrix series on the interval $[0, \omega]$.

By the algorithm (24), (25), we successively have the relations

$$C_{k+1} - C_k = -\Phi^{-1}\int_0^{\omega} [(C_k + Y_k(\tau))Q(\tau)(C_k + Y_k(\tau)) - (C_{k-1} + Y_{k-1}(\tau))Q(\tau)(C_{k-1} + Y_{k-1}(\tau)) + F(\tau, C_k + Y_k(\tau)) - F(\tau, C_{k-1} + Y_{k-1}(\tau))]\, d\tau, \tag{28}$$

$$Y_{k+1}(t) - Y_k(t) = \Phi^{-1}\int_0^{\omega} [K_A(t,\tau)(S_{k,k-1}(\tau) - S_{k-1,k-2}(\tau)) + (S_{k,k-1}(\tau) - S_{k-1,k-2}(\tau))K_B(t,\tau)]\, d\tau, \qquad k = 2, 3, \ldots \tag{29}$$

Passing to the norms in (28), we obtain

$$\begin{aligned}\|C_{k+1}-C_k\| &\le \gamma\omega(2\delta\rho+L)(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C)\\ &= q_1(\rho)(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C), \qquad k=2,3,\ldots\end{aligned} \tag{30}$$

In a similar way, for (29) we have the estimates

$$\begin{aligned}\|Y_{k+1}(t)-Y_k(t)\| &\le \|\Phi^{-1}\|\int_0^\omega[\|K_A(t,\tau)\|+\|K_B(t,\tau)\|]\|S_{k,k-1}(\tau)-S_{k-1,k-2}(\tau)\|\,d\tau\\ &\le \gamma[(\alpha+\beta)(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C)\\ &\quad+(2\delta\rho+L)(\|C_{k-1}-C_{k-2}\|+\|Y_{k-1}-Y_{k-2}\|_C)]\left(\int_0^\omega[\|K_A(t,\tau)\|+\|K_B(t,\tau)\|]\,d\tau\right)\\ &\le 0.5\gamma(\alpha+\beta)^2\omega^2(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C)\\ &\quad+0.5\gamma(\alpha+\beta)(2\delta\rho+L)\omega^2(\|C_{k-1}-C_{k-2}\|+\|Y_{k-1}-Y_{k-2}\|_C)\\ &= \tilde q_2(\rho)(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C)+(q_2(\rho)-\tilde q_2(\rho))(\|C_{k-1}-C_{k-2}\|+\|Y_{k-1}-Y_{k-2}\|_C),\end{aligned}$$

and hence we have the estimate

$$\begin{aligned}\|Y_{k+1}-Y_k\|_C &\le \tilde q_2(\rho)(\|C_k-C_{k-1}\|+\|Y_k-Y_{k-1}\|_C)\\ &\quad+(q_2(\rho)-\tilde q_2(\rho))(\|C_{k-1}-C_{k-2}\|+\|Y_{k-1}-Y_{k-2}\|_C), \quad k=2,3,\ldots,\end{aligned} \tag{31}$$

where $\tilde q_2(\rho)=0.5\gamma(\alpha+\beta)^2\omega^2$. We write the estimates (30), (31) in the matrix form

$$Z_k \le SZ_{k-1}+TZ_{k-2}, \qquad k=2,3,\ldots, \tag{32}$$

where

$$Z_m=\begin{pmatrix}\|C_{m+1}-C_m\|\\ \|Y_{m+1}(t)-Y_m(t)\|_C\end{pmatrix}, \qquad m\in\mathbb{N},$$

$$S=\begin{pmatrix}q_1(\rho) & q_1(\rho)\\ \tilde q_2(\rho) & \tilde q_2(\rho)\end{pmatrix}, \qquad T=\begin{pmatrix}0 & 0\\ q_2(\rho)-\tilde q_2(\rho) & q_2(\rho)-\tilde q_2(\rho)\end{pmatrix}.$$

Note that $S+T=Q$.

A recursion estimate of the form (32) was first obtained in the monograph [2, p. 172] when studying an algorithm for constructing a periodic solution of a second-order quasilinear vector differential equation, which could be obtained by the approach [2, Ch. 3] different from that used in the present paper. Therefore, to analyze the estimate (32), we can use the corresponding technique described in [2]. Using condition (7) and the estimates (30) and (31) [or (32)], we can prove that the sequence $((C_k,Y_k(t)))_{k=0}^\infty$ converges uniformly in $t\in[0,\omega]$ to the solution $(C,Y(t))$ of the system of matrix equations (9), (13).

Based on the estimate (32), we obtain the estimate

$$\tilde Z_k \le (E-Q)^{-1}(QZ_k+TZ_{k-1}), \qquad \tilde Z_k=\operatorname{colon}(\|C-C_k\|,\|Y-Y_k\|_C), \qquad k=1,2,\ldots,$$

which must be supplemented with estimates for Z_0 and Z_1. If $C_0=0$, $Y_0=0$, and $Y_1=0$, then these estimates can be obtained in constructive form. Indeed, by (24) we have

$$\|C_1\| \le \|\Phi^{-1}\|\left\|\int_0^\omega F(\tau,0)\,d\tau\right\| \le \|\Phi^{-1}\|\int_0^\omega\|F(\tau,0)\|\,d\tau \le \gamma\omega h.$$

Since

$$C_2 - C_1 = -\Phi^{-1} \int_0^\omega [C_1 Q(\tau) C_1 + F(\tau, C_1) - F(\tau, 0)]\, d\tau,$$

we have the inequality

$$\|C_2 - C_1\| \le \gamma \int_0^\omega [\|C_1\| \|Q(\tau)\| \|C_1\| + \|F(\tau, C_1) - F(\tau, 0)\|]\, d\tau \le \gamma\omega(\delta\rho^2 + L\rho).$$

Further, by relation (25) we have

$$Y_2(t) = \Phi^{-1} \int_0^\omega [K_A(t,\tau) S_{1,0}(\tau) + S_{1,0}(\tau) K_B(t,\tau)]\, d\tau. \tag{33}$$

Carrying out the estimates in (33), we obtain

$$\begin{aligned} \|Y_2\|_C &\le \|\Phi^{-1}\| \int_0^\omega (\|K_A(t,\tau)\| + \|K_B(t,\tau)\|) \|S_{1,0}(\tau)\|\, d\tau \\ &\le \gamma \int_0^\omega [(\alpha+\beta)\|C_1\| + \|F(\tau,0)\|](\|K_A(t,\tau)\| + \|K_B(t,\tau)\|)\, d\tau \\ &\le 0.5\gamma(\alpha+\beta)\omega^2[(\alpha+\beta)\|C_1\| + h] \le 0.5\gamma(\alpha+\beta)\omega^2[\gamma(\alpha+\beta)\omega + 1]h. \end{aligned}$$

Thus, we have proved the following assertion.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Then there exists a unique solution of problem* (1), (2) *in the domain* D_ρ. *This solution can be represented as the limit of a uniformly converging sequence of matrix functions defined by the recursion integral relations* (24) *and* (25) *and satisfying conditions* (22) *and* (23).

This theorem substantially improves Theorem 1, but the algorithm (24), (25) contains fairly simple computational operations and hence is convenient in applications.

Remark 3. Obviously, the choice of the matrices C_0, Y_0 and C_1, Y_1 is somewhat ambiguous and is constrained only by conditions (22) and (23) occurring in the derivation of the matrices C_2 and Y_2.

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