УДК 517.91

ON NEW PhD STUDENTS' RESULTS IN QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

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This article presents some new PhD students' results in qualitative theory of differential equations. Previous reviews on this topic can be found in [1, 2].

1. On the behavior of solutions to differential equations with general power-law nonlinearities (T. Korchemkina) [3–5]

For the equation

$$y'' = p(x, y, y')\,|y|^{k_0}|y'|^{k_1}\,\mathrm{sgn}\,(y\,y'), \qquad k_0, k_1 > 0, \tag{1}$$

with positive continuous in x and Lipschitz continuous in u, v function $p(x, u, v)$, the qualitative and asymptotic behavior of solutions is studied. Results on the qualitative behavior of solution depending on nonlinearity exponents are presented in [3]. The asymptotic behavior of solutions unbounded near their domain boundaries is described both in the cases of constant and non-constant potential in [4].

Problem 1. Compare the asymptotic behavior of solutions to equation (1) for the cases $k_1 > 0$ and $k_1 = 0$ (see [6, Ch. V], [7]).

Theorem 1.1 Suppose $k_0, k_1 > 0$. Let $p(x, u, v)$ be a positive continuous in x and Lipschitz continuous in u, v function. Then the set of all maximally extended solutions to equation (1) can be divided into the following five types according to their behavior:

0) *constant solutions,*

1) *increasing positive solutions,*

2) *increasing negative solutions,*

3) *increasing solutions negative near the left boundary of their domain and positive near the right one,*

4) *decreasing solutions positive near the left boundary of their domain and negative near the right one.*

Put $$\alpha = \frac{2-k_1}{k_0+k_1-1}, \qquad C(P) = \left(\frac{|\alpha|^{1-k_1}\,|\alpha+1|}{P}\right)^{\frac{1}{k_0+k_1-1}},$$

$$D(P) = \left(\frac{|\alpha|^{k_0}|\alpha + 1|}{P}\right)^{\frac{1}{k_0+k_1-1}}, \qquad P \in \mathbb{R}.$$

Theorem 1.2 *Suppose* $k_0 + k_1 > 1, k_1 < 2$ *(hence* $\alpha > 0$*), and the function* $p(x,u,v)$ *is positive, continuous in* x*, Lipschitz continuous in* u, v *and has a positive limit* P_a *as* $x \to a,\ u \to +\infty,\ v \to +\infty$ *for any* $a \in \mathbb{R}$*. Then every increasing solution to equation* (1) *has a vertical asymptote* $x = x^*$ *with the following asymptotic behavior near the right boundary of its domain*:

$$y(x) = C(P_{x^*})\,(x^* - x)^{-\alpha}\,\big(1 + o(1)\big), \qquad x \to x^* - 0,$$

$$y'(x) = D(P_{x^*})\,(x^* - x)^{-\alpha-1}\,\big(1 + o(1)\big), \qquad x \to x^* - 0.$$

Put $C_0(s,t) = (s\,|1 - k_1|)^{\frac{1}{1-k_1}}\,|t|^{\frac{k_0}{1-k_1}}$.

Theorem 1.3 *Suppose* $k_1 > 2$*. Let* $y(x)$ *be an increasing solution to equation (1), let* $x^* < +\infty$ *be its domain's right boundary. Put* $y^* = \lim\limits_{x \to x^*-0} y(x)$ *and let* $p(x,u,v) \to p^*$ *as* $x \to x^*, u \to y^*, v \to +\infty$*. Then*

$$y'(x) = C_0(p^*, y^*)\,(x^* - x)^{\frac{1}{k_1-1}}\big(1 + o(1)\big), \qquad x \to x^* - 0.$$

For the equation

$$y''' = p(x,y,y',y'')\,|y|^{k_0}|y'|^{k_1}|y''|^{k_2}\,\mathrm{sgn}\,(y\,y'\,y''), \quad k_0, k_1, k_2 > 0, \tag{2}$$

with the positive continuous and Lipschitz continuous in u, v, w function $p(x,u,v,w)$, the qualitative behavior of solutions with positive initial data depending on the values of k_0, k_1, k_2 is studied in [5].

Problem 2. Compare the asymptotic behavior of solutions to equation (2) for the cases $k_0, k_1, k_2 > 0$ and $k_1 = k_2 = 0$ (see [8]).

Theorem 1.4 Suppose $k_0 + k_1 + k_2 \neq 1$, $k_2 \neq 1$, $k_2 \neq 2$, *and the function* $p(x,u,v,w)$ *is positive, continuous and Lipschitz continuous in* u, v, w*. Let* $y(x)$ *be a maximally extended solution to equation (2) satisfying the conditions* $y(x_0) \geq 0, y'(x_0) \geq 0, y''(x_0) > 0$ *at some point* x_0*. Then*

1. $y \to +\infty, y' \to +\infty, y'' \to +\infty$ *as* $x \to x^* = +\infty$ *or*
   $y \to +\infty, y' \to +\infty, y'' \to +\infty$ *as* $x \to x^* < +\infty$ *if* $k_0 + k_1 + k_2 < 1$;

2. $y \to +\infty, y' \to +\infty, y'' \to +\infty$ *as* $x \to x^* < +\infty$
   if $k_0 + k_1 + k_2 > 1, k_1 \leq 1, k_2 < 1$;

3. $y \to +\infty, y' \to +\infty, y'' \to +\infty$ *as* $x \to x^* < +\infty$ *or*
$y \to \text{const}, y' \to +\infty,\ y'' \to +\infty$ *as* $x \to x^* < \infty$
if $k_1 > 1, k_2 < 1$;

4. $y \to \text{const}, y' \to +\infty,\ y'' \to +\infty$ *as* $x \to x^* < \infty$ *if* $1 < k_2 < 2$;

5. $y \to \text{const}, y' \to \text{const},\ y'' \to +\infty$ *as* $x \to x^* < \infty$ *if* $k_2 > 2$.

2. Sturm-type theorems for high-order nonlinear differential equations (V. Rogachev) [12, 15–17]

For the equation

$$y^{(n)} + p\left(x, y, y', \dots, y^{(n-1)}\right) |y|^k \operatorname{sgn}(y) = 0 \tag{3}$$

with $n \geq 3$, $k > 0$, $k \neq 1$, a continuous in $x, \xi_1, \dots, \xi_n$ and Lipshitz continuous in $\xi_1, \dots, \xi_n$ function $p(x, \xi_1, \dots, \xi_n)$ satisfying the inequalities

$$0 < m \leq \mathrm{p}(x, \xi_1, \dots, \xi_n) \leq M < \infty,$$

the qualitative behavior of solutions is studied. Equation (3) can be considered as a nonlinear generalization of a linear equation

$$y^{(n)} + q(x)\, y = 0 \tag{4}$$

with continuous q.

Theorem (Sturm J. Ch. F. [9]). *For $n = 2$, if there are two consecutive zeroes of a solution to (4), then there is exactly one zero of any other linearly-independent solution between them.*

Theorem (Kondrat'ev V. A. [10, 11]). *For $n = 3$ and $q > 0$ (or $q < 0$), if there are two consecutive zeros of a solution to (4), then there can be no more than 2 zeros of any other solution between them. For $n = 4$ and $q > 0$, there is no more than 4 zeros. For $n = 4$ and $q < 0$, there is no more than 3 zeros. For $n > 4$ and $q > 0$, it is possible to have a solution with any number of zeros between two consecutive zeros of another solution.*

Problem 3. To obtain analog of these theorems for nonlinear equations with power-law nonlinearities.

For equation (3), using methods from [12–14], we obtained the following results.

Theorem 2.1 *For any integer $s \geq 2$ and any segment $[a, b]$ there is a solution to (3) defined on $[a, b]$, vanishing at its end-points a, b, and having exactly s zeros on $[a, b]$.*

Theorem 2.2 *For $k > 1$ and any segment $[a, b)$ there is a solution to (3) defined on $[a, b)$, vanishing at the point a and having a countable infinite number of zeros on $[a, b)$.*

Theorem 2.3 *For* $0 < k < 1$, *any positive constant* p_0 *and any segment* $[a, b]$ *there is a defined on* $[a, b]$ *solution to (3), with* $p(x, \xi_1, \dots, \xi_n) \equiv p_0$, *vanishing on its end-points a, b, and having a countable infinite number of zeros on* $[a, b]$. *Besides, there is a non-trivial solution defined on the segment* $[a, b]$, *vanishing on its end-points a, b, and having an uncountable number of zeros on* $[a, b]$.

Those theorems extrapolate Kondratiev's results to the case of non-linear higher-order ($n \geq 3$) equations with positive potential *p*. In the nonlinear case, any equation has a solution with any (and not only finite) number of zeros between two consecutive zeros of another solution, regardless of *n*.

For the case of negative p, we can obtain similar results if *n* is odd (see [15–17]). In the case of even *n* and negative *p*, the problem is still open, although it is proved that for *n* = 4 and $p(x, \xi_1, \dots, \xi_n) \equiv p_0 < 0$, equation (3) also has a solution with a given finite number of zeros on [*a*, *b*] (see [8]).

This work is partially supported by Grant RFBR 19-31-90168.

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