On two-point boundary value problem for the matrix Riccati equation with parameter

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Consider a Riccati equation of the following form:

$$\frac{dX}{dt} = \lambda(A(t)X + XB(t) + XQ(t)X + F(t)) \equiv G(t, X, \lambda), \qquad (1)$$

where $A, B, Q, F \in \mathbb{C}(I, \mathbb{R}^{n\times n})$, $I = [0, \omega]$, $\omega > 0$, $\lambda \in \mathbb{R}$.

We study a two-point boundary-value problem for (1) in case of

$$MX(0, \lambda) + NX(\omega, \lambda) = 0, \qquad (2)$$

where M and N are real $n \times n$ matrices.

Equation (1) is prominent in the differential equation theory and its applications [1–9]. Similar problems were considered with the aid of qualitative methods in [1, 4–7] and on the basis of constructive methods in [3, 8–11].

The present work is a continuation of [10] and deals with a constructive analysis of problem (1), (2) on the basis of the method presented in [3, ch. 1].

We studied the issues of convergence, the rate of convergence of the algorithm (4).

We introduce the notations:

$$D_\rho = \{(t, X) : t \in I, \|X\| \le \rho\},\ \alpha = \max_{t\in I} \|A(t)\|,\ \beta = \max_{t\in I} \|B(t)\|,$$

$$h = \max_{t\in I} \|F(t)\|,\ \delta = \max_{t\in I} \|Q(t)\|,\ \varepsilon = |\lambda|, \mu = \max(\|M\|, \|N\|),$$

$$\gamma = \|(M+N)^{-1}\|,\ \|X\|_{\mathbb{C}} = \max_{t\in I} \|X(t,\lambda)\|,$$

$$\varepsilon_1 = \frac{\rho}{a_0\rho^2 + a_1\rho + a_2},\ \varepsilon_2 = \frac{1}{2a_0\rho + a_1},\ \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\},$$

$$a_0 = \gamma\mu\delta\omega,\ a_1 = \gamma\mu(\alpha+\beta)\omega,\ a_2 = \gamma\mu\omega h,\ q = \varepsilon(2a_0\rho + a_1),$$

where $\rho > 0$, $\|X\|_{\mathbb{C}}$ is the norm in finite–dimensional Banach algebra $\mathcal{B}(n)$ of continuous $n \times n$ matrices-functions; $\|\centerdot\|$ is the corresponding norm of matrixes, for example, any of norms given in [11, p. 21].

The problem (1), (2) is equivalent to the matrix integral equation

$$X(t,\lambda) = (M+N)^{-1}\times$$
$$\times\left\{M\int_0^t G(\tau, X(\tau,\lambda), \lambda)d\tau - N\int_t^\omega G(\tau, X(\tau,\lambda), \lambda)d\tau\right\}. \tag{3}$$

Theorem. *Let* $\det(M+N) \neq 0$. *Then for* $|\lambda| < \varepsilon_0$ *the solution of problem (1), (2) exists in the region* D_ρ, *it is unique and can be presented as a uniform limit of a sequence of matrix functions determined by the recurrent integrated relationship*

$$X_{k+1}(t,\lambda) = (M+N)^{-1}\left\{M\int_0^t G(\tau, X_k(\tau,\lambda), \lambda)d\tau - \right.$$
$$\left. -N\int_t^\omega G(\tau, X_k(\tau,\lambda), \lambda)d\tau\right\},\ k = 0, 1, 2, \ldots, \tag{4}$$

where $X_0(t,\lambda)$ is arbitrary $\mathbb{C}(I\times\mathbb{R},\mathbb{R}^{n\times n})$-class matrix, which belongs to the sphere $\|X_0\|_{\mathbb{C}} \le \rho$. Herewith the matrix functions $X_m(t,\lambda)$ $(m = 1, 2, \dots)$ it also satisfy to condition (2).

By using induction for k, one can show readily that members of sequence $\{X_k(t,\lambda)\}_0^\infty$, belong to the sphere $\|X\|_{\mathbb{C}} \le \rho$. The sequence converges uniformly to the solution of (3).

Also received estimates

$$\|X - X_k\|_{\mathbb{C}} \le \frac{q^k}{1-q}\|X_1 - X_0\|_{\mathbb{C}} \;\; k = 0, 1, 2, \dots. \tag{5}$$

We have from (5) for $k = 0, X_0 = 0$

$$\|X\|_{\mathbb{C}} \le \frac{\|X_1\|_{\mathbb{C}}}{1-q}. \tag{6}$$

From (4) for $X_0 = 0$ we obtain an estimate for X_1:

$$\|X_1\|_{\mathbb{C}} \le \gamma\mu\omega\varepsilon h. \tag{7}$$

Using (7) and (6) we have

$$\|X\|_{\mathbb{C}} \le \frac{\gamma\mu\omega\varepsilon h}{1-q}.$$

Remark. The conditions for the unique solvability of problem (1),(2) are expressed in terms of its initial data. Algorithm (4) contains simple computational operations and is therefore convenient for possible applications. The corresponding estimates are obtained in terms of the problem.

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