Теорема 5 [5]. *Если выполнено одно из условий*

$$\overline{\lim_{t\to+\infty}} \int\limits_t^{+\infty} \int\limits_0^t d_\tau r(s,\tau)\,ds > 1, \quad \underline{\lim_{t\to+\infty}} \int\limits_t^{+\infty} \int\limits_0^t d_\tau r(s,\tau)\,ds > 1/e,$$

то все решения уравнения (3) *устойчивого типа осциллируют.*

Работа выполнена при финансовой поддержке при поддержке Минобрнауки РФ (госзадание FSNM-2023-0005).

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ON ASYMPTOTIC EQUIVALENCE OF HIGHER-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

I. V. Astashova

We study the problem of asymptotic equivalence of the equations

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)\,|y(x)|^k \operatorname{sgn} y(x) = f(x) \tag{1}$$

and

$$z^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)\,|z(x)|^k \operatorname{sgn} z(x) = 0 \tag{2}$$

with $n \ge 2$, $k > 1$, and continuous functions $p(x)$, $f(x)$ and $a_j(x)$. Equation (2) is a so-called Emden–Fowler type differential equation. It was considered from different points of view (see for example [1, 2] and the bibliography there). In particular, the asymptotic behavior of its solutions vanishing at infinity is described. (See also [3–6].) So, if an asymptotic equivalence of equations (1) and (2) exists, it is possible to describe the asymptotic behavior of vanishing at infinity solutions to equation (1), too. Previous results are formulated in [7–10]. The asymptotic equivalence of ordinary differential equations and their systems can be useful to investigate some problems for partial differential equations (see, for example, [11]). Note that the notion of asymptotic equivalence can be used in different senses (cf.[10, 12–19]).

Hereafter we denote $|y|^k \operatorname{sgn} y$ by $[y]^k_\pm$.

Theorem 1. *Let* $a_0, \ldots, a_{n-1}$, f, g, *and* p *be continuous functions defined in a neighborhood of* ∞. *Suppose that* p, f, *and* g *are bounded while* $a_0, \ldots, a_{n-1}$ *satisfy the inequalities*

$$\int_{x_0}^{\infty} x^{n-j-1}|a_j(x)|\,dx < \infty,\ j \in \{0, \ldots, n-1\}. \tag{3}$$

If y *is a solution to the equation*

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)} + p(x)[y(x)]_{\pm}^{k} = f(x)\, e^{-\gamma x} \tag{4}$$

with $n \geq 2$, $k > 1$, $\gamma > 0$ *and* $y(x) \to 0$ *as* $x \to +\infty$, *then there exists a unique solution* z *to the equation*

$$z^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) z^{(j)}(x) + p(x)[z(x)]_{\pm}^{k} = g(x)\, e^{-\gamma x} \tag{5}$$

such that $|z(x) - y(x)| = O(e^{-\gamma x})$ *as* $x \to +\infty$.

To prove this theorem we need the following lemmas.

Lemma 1. *Any linear differential operator*

$$L : y \mapsto y^{(n)} + \sum_{j=0}^{n-1} a_j y^{(j)} \tag{6}$$

with all continuous functions $a_j(x)$ *satisfying (3) can be represented in a neighbourhood of* $+\infty$ *as the composition operator*

$$L = D_b = b_0 B_1 \circ \cdots \circ B_n,$$

where $b = (b_0, b_1, \ldots, b_n)$, *all* B_j, $j = 1, \ldots, n$, *are the first-order operators* $u \mapsto (b_j u)'$ *and each* b_j, $j = 0, \ldots, n$, *is a* $\mathcal{C}^j$ *function satisfying at infinity the following conditions:*

(i) $b_j(x) \to 1$,

(ii) $x^i b_j^{(i)}(x) \to 0$ *for all* $i \in \{1, \ldots, j-1\}$,

(iii) $\to {}_{x_0} x^{i-1} \left| b_j^{(i)}(x) \right| dx < \infty$ *for all* $i \in \{1, \ldots, j\}$ *and some* $x_0 \in \mathbb{R}$.

Now, for $b = (b_0, b_1, \ldots, b_n)$ and $j \in \{0, \ldots, n\}$ put

$$b - j = (b_j, \ldots, b_n).$$

Note that if a tuple b satisfies the conditions from Lemma 1, then so does the tuple $b - j$.

Lemma 2. *Let* $b = (b_0, b_1, \ldots, b_n)$ *satisfy the conditions from Lemma 1. If a function* y *satisfies at infinity both* $y \to 0$ *and* $D_b(y) \to 0$, *then the same is true for all functions* $D_{b-j}(y)$, $0 < j < n$.

Lemma 3. *Suppose a tuple* $b = (b_0, b_1, \ldots, b_n)$ *satisfies the conditions from Lemma 1 and a function* y *satisfies, on a segment* I *of length* Δ, *the inequality* $|D_{b-j}(y)| \geq W$ *with some* $j \in \{1, \ldots, n\}$ *and a constant* $W > 0$. *Then there exists a segment* $I' \subset I$ *of length* $4^{j-n}\Delta$ *with* $|y(x)| \geq (2^{j-n}\,\beta)^{n+1-j}\, W \Delta^{n-j}$ *for all* $x \in I'$.

Now we can formulate the following

Corollary. *Under the conditions of Theorem 1, a function y is a solution to equation (4) tending to zero as $x \to +\infty$ if and only if*

$$b_n y = (J_{n-1} \circ \cdots \circ J_0)\Big[e^{-\gamma x} f(x) - p(x)[y(x)]_{\pm}^{k}\Big], \tag{7}$$

where the operators J_j take each sufficiently rapidly decreasing continuous function φ to the vanishing at infinity primitive function of φ/b_j :

$$J_j[\varphi](x) = -\int_x^{\infty} \frac{\varphi(\xi)}{b_j(\xi)} d\xi.$$

From Theorem 1 we can obtain

Theorem 2. *Suppose that the function $f(x)$ in equation (1) is continuous and satisfies the condition*

$$|f(x)| \le C e^{-\gamma x},\ C > 0,\ \gamma > 0, \tag{8}$$

all $a_0, \ldots, a_{n-1}$ are continuous functions satisfying (3), and $p(x)$ is a bounded continuous function.

Then for any solution $y(x)$ to equation (1) tending to zero as $x \to \infty$, there exists a solution $z(x)$ to equation (2) such that

$$|y(x) - z(x)| = O(e^{-\gamma x}), \quad x \to \infty. \tag{9}$$

Similarly, for any solution $z(x)$ to equation (2) tending to zero as $x \to \infty$, there exists a solution $y(x)$ to equation (1) satisfying (9).

Remark. *Note that a similar result is true for equation (1) with a power-law small right-hand side. (For the case $a_j = 0$ see[9].)*

The work is partially supported by RSF (Project 20-11-20272).

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A FORMULA FOR THE CENTRAL EXPONENT OF DISCRETE TIME-VARYING SYSTEMS

Adam Czornik and Michał Niezabitowski

Since the famous work of A.M. Lyapunov [1], Lyapunov exponents entered the canon of dynamical systems theory and are, along with other numerical characteristics such as Bohl or Perron exponents, commonly used tools to describe properties of dynamical systems. One of the problems with numerical calculation of Lyapunov exponents of linear systems with variable coefficients is their discontinuity as a function of coefficients of the system. This property was already noticed by O. Perron in [2]. For this reason, many works in the literature concern the description of possible changes in the Lyapunov exponents under the influence of various kinds of parametric perturbations. A summary of this work for continuous-time systems can be found in the monograph [3], and for discrete-time systems in the monograph [4]. In particular, it was shown that the maximum upward shift of the largest Lyapunov exponent is described by the so-called central exponent (see [5]) and Theorem 1 below). The significance of this exponent for the theory of stability lies in the fact that its negativity implies, inter alia, the exponential stability of the perturbed system for all parametric perturbation tending to zero. However, this exponent is expressed by the transition matrix of the unperturbed system and therefore, in general, it is difficult to compute. Additionally, the central exponent itself is generally not a continuous function of the coefficients but only a semi-continuous function from above (see Chapter 4 in [3]).

On the other hand, for time-invariant systems, a comprehensive description of the dynamic properties can be obtained through the spectrum of the system matrix. This brings to mind an attempt to express the numerical characteristics of systems with variable coefficients through the eigenvalues of the matrix of coefficients. In the general case, it is unfortunately impossible, because there are examples of exponentially uniformly stable continuous systems, whose coefficient matrices have spectra lying in the right half-plane, as well as examples of unstable systems with coefficient matrices with only eigenvalues with a negative real part (see e.g. [6]