10. Astashova I., Bartusek M., Dosla Z., Marini M. *Asymptotic proximity to higher order nonlinear differential equations* // Advances in Nonlinear Analysis. **11**(1). 2022. 1598–1613.

11. Egorov Yu. V., Kondratiev V. A., Oleinik O. A., *Asymptotic behavior of the solutions to nonlinear elliptic and parabolic systems in tube domains* // Mat. Sbornik. **189**:3. 1998. 45-68. (Russian)

12. Brower F., Wong J. S. W., *Asymptotic relationships between solutions of two systems of ordinary differential equations* // J. Diff. Equations. 1969. **6**. 527-543.

13. Svec M., *Asymptotic relationship between solutions of two systems of differential equations*// Czechoslovak Math. J. 29. 1974. P. 44–58.

14. Saito S., *Asymptotic equivalence of quasilinear ordinary differential systems* // Math. Japan. 37. 1992. 503-513.

15. Choi S. K., Goo Y. H., Koo N. J., *Asymptotic equivalence between two linear differential systems* // Ann. Differential Equations. 13. 1997. P. 44–52.

16. Zafer A., *On asymptotic equivalence of linear and quasilinear difference equations* // Appl. Anal. 2005 **84** (9). 899-908.

17. Pinto M., *Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments* // Mathematical and Computer Modelling. 2009. **49**. 9–10. 1750–1758.

18. Reinfelds A., *Asymptotic equivalence of difference equations in Bahach Space* // Theory and Applications of Difference Equations and Discrete Dynamical Systems. Springer. 2014. P. 215-222.

19. Samoilenko A., Stanzhytskyi O., *Qualitative and Asymptotic Analysis of Differential Equations with Random Perturbations* / World Scientific. **78**. 2011.

A FORMULA FOR THE CENTRAL EXPONENT OF DISCRETE TIME-VARYING SYSTEMS

Adam Czornik and Michał Niezabitowski

Since the famous work of A.M. Lyapunov [1], Lyapunov exponents entered the canon of dynamical systems theory and are, along with other numerical characteristics such as Bohl or Perron exponents, commonly used tools to describe properties of dynamical systems. One of the problems with numerical calculation of Lyapunov exponents of linear systems with variable coefficients is their discontinuity as a function of coefficients of the system. This property was already noticed by O. Perron in [2]. For this reason, many works in the literature concern the description of possible changes in the Lyapunov exponents under the influence of various kinds of parametric perturbations. A summary of this work for continuous-time systems can be found in the monograph [3], and for discrete-time systems in the monograph [4]. In particular, it was shown that the maximum upward shift of the largest Lyapunov exponent is described by the so-called central exponent (see [5]) and Theorem 1 below). The significance of this exponent for the theory of stability lies in the fact that its negativity implies, inter alia, the exponential stability of the perturbed system for all parametric perturbation tending to zero. However, this exponent is expressed by the transition matrix of the unperturbed system and therefore, in general, it is difficult to compute. Additionally, the central exponent itself is generally not a continuous function of the coefficients but only a semi-continuous function from above (see Chapter 4 in [3]).

On the other hand, for time-invariant systems, a comprehensive description of the dynamic properties can be obtained through the spectrum of the system matrix. This brings to mind an attempt to express the numerical characteristics of systems with variable coefficients through the eigenvalues of the matrix of coefficients. In the general case, it is unfortunately impossible, because there are examples of exponentially uniformly stable continuous systems, whose coefficient matrices have spectra lying in the right half-plane, as well as examples of unstable systems with coefficient matrices with only eigenvalues with a negative real part (see e.g. [6]

p. 257). There are also analogous examples for discrete systems. It turns out, however, that if the coefficients of the system change slowly enough, then from the location of the spectra of the coefficient matrix, certain properties concerning the asymptotic properties of solutions can be deduced. This is the basic idea behind the so-called 'freezing method' initiated by Desoer's work [7]. A summary of the results obtained using this technique can be found in Section 10.1 of [8].

In this work we deal with the relationship between the largest in absolute value eigenvalues of the coefficient matrix and central exponent of a discrete time-varying system. The main result of the work states that for systems with slowly varying coefficients (see Definition 1) the central exponent is the upper limit of arithmetic-mean of the logarithms of the largest module of eigenvalues of the coefficient matrix. Systems with slowly varying coefficients have been recently considered in [9] where formulae for the largest and smallest Bohl exponents were obtained. The work also contains a numerical example illustrating the obtained result.

Let $\|x\|$ be the Euclidean norm of $x \in \mathbb{R}^d$ and $\|C\|$ be the operator norm induced by Euclidean norm of a matrix $C \in \mathbb{R}^{d\times d}$. A Lyapunov sequence is a sequence $C = (C(n))_{n\in\mathbb{N}}$ of invertible square matrices such that

$$\max\{\|C\|_\infty, \|C^{-1}\|_\infty\} < \infty,$$

where $\|C\|_\infty := \sup_{n\in\mathbb{N}} \|C(n)\|$.

We will consider systems of the following form

$$x(n+1) = A(n)x(n),\ n \in \mathbb{N}, \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}}$ is a Lyapunov sequence. Let us denote by $(\Phi_A(n,m))_{n,m\in\mathbb{N}}$ the transition matrix of system (1).

Definition 1. The Lyapunov exponent λ_A of system (1) is defined as follows

$$\lambda_A = \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_A(n,0)\|.$$

Consider now a perturbed system

$$z(n+1) = (A(n) + Q(n))\, z(n),\ n \in \mathbb{N}, \tag{2}$$

where the perturbation $Q = (Q(n))_{n\in\mathbb{N}}$ is a bounded sequence of d by d matrices such that $A + Q = (A(n) + Q(n))_{n\in\mathbb{N}}$ is a Lyapunov sequence. It is clear that for each Lyapunov sequence A there exists a $\delta_A > 0$ such that $A + Q$ is a Lyapunov sequence for each Q such that $\|Q\|_\infty < \delta_A$. Considering the perturbed system (2) we will always assume that Q is such that $\|Q\|_\infty < \delta_A$.

Consider the following two quantities

$$\Omega_1(A) = \lim_{q\to 0^+} \left(\sup_{\|Q\|_\infty < q} \lambda_{A+Q} \right),$$

$$\Omega_2(A) = \sup\left\{ \lambda_{A+Q} : \lim_{n\to\infty} Q(n) = 0 \right\}.$$

The quantity $\Omega_1(A)$ introduced for continuous-time systems by R. E. Vinograd in articles [10] and [11], in which he obtained an upper bound of $\Omega_1(A)$ in terms of the transition Cauchy matrix of the system. V. M. Millionshchikov proved in paper [12] that the upper bound obtained by R. E. Vinograd is sharp. The problem of calculating $\Omega_1(A)$ and $\Omega_2(A)$ for discrete-time systems was investigated in [5], where the following theorem has been proved.

Theorem 1. *The following equality holds*

$$\Omega_1(A) = \Omega_2(A) = \Omega_C(A),$$

where

$$\Omega_C(A) = \lim_{N\to\infty} \frac{1}{N} \left(\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \|\Phi_A(k+N,k)\| \right).$$

The number $\Omega_C(A)$ is called the central exponent of system (1).

The main objective of this paper is to provide a formula for $\Omega_C(A)$ in the terms of eigenvalues of matrices $A(n)$ under certain additional assumption about the sequence A. In the next theorem $C \in \mathbb{R}^{d\times d}$, $\lambda(C)$ is the greatest absolute value of the eigenvalues of matrix C.

Theorem 2. *If*

$$\lim_{n\to\infty} \|A(n+1) - A(n)\| = 0,$$

then

$$\Omega_C(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \lambda(A(i)).$$

Refrences

1. Lyapunov A.M.*The general problem of the stability of motion,* // Internat. J. Control. 1992. Vol. 55, № 3, P. 531–773.

2. Perron O. *Die Ordnungszahlen linearer Differentialgleichungssysteme.* // Math. Z. 1930. Vol. 31, № 1, P. 748–766.

3. Izobov N. A. *Lyapunov Exponents and Stability*, Cambridge, Cambridge Scientific Publishers, 2012.

4. Czornik A. *Perturbation Theory for Lyapunov Exponents of Discrete Linear Systems*, Kraków, Wydawnictwo AGH, 2012.

5. A. Babiarz A., Barabanov E, Czornik A, Niezabitowski M. Vaidzelevich A.*On the influence of small perturbations on the senior Lyapunov exponent of the discrete time-varying system.* // IFAC-Papers On Line. 2017. Vol. 50(1). P. 7462–7466.

6. Hinrichsen D. and Prichard A. J. *Mathematical systems theory I.* Berlin, Springer-Verlag, 2005.

7. Desoer C. *Slowly varying discrete system* $x_{i+1} = A_i x_i$. // Electronics Letters. 1970. Vol. 6. № 11. P. 339–340.

8. Gil M.I. *Difference Equations in Normed Spaces. Stability and Oscillations.* North-Holland Mathematics Studies, Vol. 206. Amsterdam, Elsevier, 2007.

9. Czornik A., Simek K. *A formula for the lower Bohl exponent of discrete time-varying systems,* acepted for publication in Archives of Control Sciences.

10. Vinograd R. *Estimate of the jump of a characteristic exponents for small perturbations.* // Dokl. Akad. Nauk SSSR, 1957. Vol. 114. № 3. P. 459–461.

11. Vinograd R. *On the central characteristic exponent of a system of differential equations.* // Mat. Sb. 1957. Vol. 42, № 2. P. 207–222.

12. Millionshchikov, V. (1969). A proof of the attainability of the central exponents of linear systems. Sibirskij Matematicheskij Zhurnal (Siberian Mathematical Journal), 1969. Vol. 10. № 1, 99–104 (69–73).