THE METHOD OF FIELD CHARACTERISTICS FOR AUTONOMOUS SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

D. S. Zhalukevich

Let us consider an autonomous system of the 2-nd order [1]:

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad (x,y) \in G \subseteq \Re^2. \tag{1}$$

The matrix of the system (1) has the form

$$A_1 = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}, \tag{2}$$

the determinant of the matrix (3) will be equal to

$$J_1 = detA_1 = P_xQ_y - P_yQ_x. \tag{3}$$

We introduce the notation

$$\vec{v} = \hat{T}\vec{r}, \quad \vec{a} = \hat{T}\vec{v}, \quad \vec{w} = \hat{T}\vec{a}, \tag{4}$$

where $\vec{r} = x\vec{i} + y\vec{j}$, $\hat{T} = \dfrac{d}{dt} = \vec{v} \cdot \nabla$, $\nabla = \dfrac{\partial}{\partial x}\vec{i} + \dfrac{\partial}{\partial y}\vec{j}$.
Differentiating the system (1) in time, we get the system

$$\ddot{x} = F_1(x,y,\dot{x},\dot{y}), \quad \ddot{y} = F_2(x,y,\dot{x},\dot{y}), \tag{5}$$

where

$$F_1 = \dot{P} = \vec{v} \cdot \nabla P, \quad F_2 = \dot{Q} = \vec{v} \cdot \nabla Q. \tag{6}$$

Differentiating the system (5) in time, we get the system

$$\dddot{x} = \Phi_1(x,y,\dot{x},\dot{y},\ddot{x},\ddot{y}), \quad \dddot{y} = \Phi_2(x,y,\dot{x},\dot{y},\ddot{x},\ddot{y}), \tag{7}$$

where

$$\Phi_1 = \dot{F_1} = \ddot{P} = \vec{a} \cdot \nabla P + \vec{v} \cdot \nabla F_1, \quad \Phi_2 = \dot{F_2} = \ddot{Q} = \vec{a} \cdot \nabla Q + \vec{v} \cdot \nabla F_2. \tag{8}$$

The determinant of an arbitrary order can be written through Poisson brackets

$$J_k = detA_k = -\{P^{(k-1)}, Q^{(k-1)}\}, \tag{9}$$

where

$$P^{(0)} = P, \quad Q^{(0)} = Q, \qquad P^{(k-1)} = \frac{d^{(k-1)}P}{dt^{(k-1)}}, \quad Q^{(k-1)} = \frac{d^{(k-1)}Q}{dt^{(k-1)}}, \quad k = \overline{1,n}.$$

Definition 1. *The matrix of system (1) is called primary, system (5) secondary and system (7) tertiary.*

Theorem 1. *For a stationary plane field of the system (1), the connections between the primary, secondary and tertiary matrices have the form*

$$A_2 = A_1^2 + \vec{v} \cdot \nabla A_1, \tag{10}$$

$$A_3 = A_1A_2 + A_2A_1 + \vec{v} \cdot \nabla A_2 + \vec{a} \cdot \nabla A_1. \tag{11}$$

It can be seen from (10) and (11) that at the equilibrium point $\vec{v}_0 = 0$ and $\vec{a}_0 = 0$ the determinants of the secondary and tertiary matrices will be equal $J_2 = J_1^2$, $J_3 = 2J_1J_2$. Let's introduce the notation [2]

$$\sigma_1 = TrA_1 = \nabla \cdot \vec{v}, \quad \sigma_2 = TrA_2 = \nabla \cdot \vec{a}, \quad \sigma_3 = TrA_3 = \nabla \cdot \vec{w}, \tag{12}$$

$$\vec{\rho}_1 = \nabla \times \vec{v} = \rho_1 \vec{k}, \quad \vec{\rho}_2 = \nabla \times \vec{a} = \rho_2 \vec{k}, \quad \vec{\rho}_3 = \nabla \times \vec{w} = \rho_3 \vec{k}, \tag{13}$$

where TrA_1, TrA_2 and TrA_3 are traces of the primary, secondary and tertiary matrices. The expression σ_1 is called the continuity equation, and $\vec{\rho}_1$ the velocity vortex (vorticity) [3]. Flows and circulations of arbitrary order are related by expressions

$$\sigma_{k+1} = \sigma_k^{(1)} = \sigma_1^{(k)}, \quad \rho_{k+1} = \rho_k^{(1)} = \rho_1^{(k)}, \tag{14}$$

where

$$\sigma_1^{(k)} = \frac{d^{(k)}\sigma_1}{dt^{(k)}}, \quad \rho_1^{(k)} = \frac{d^{(k)}\rho_1}{dt^{(k)}}, \quad k = \overline{1,n}.$$

The characteristic equations for the primary, secondary and tertiary matrices have the form

$$\lambda_{kl}^2 - \sigma_k \lambda_{kl} + J_k = 0, \quad D_k = \sigma_k^2 - 4J_k, \tag{15}$$

where

$$k = \overline{1,3}, \quad l = \overline{1,2}.$$

Definition 2. *The values $\sigma_k, J_k, D_k, \rho_k$ are called for primary, secondary and tertiary characteristics of a stationary plane field of the system (1).*

Theorem 2. *For a stationary plane field of the system (1), the relations between the primary, secondary and tertiary characteristics of the field have the form [2]*

$$\sigma_2 = \sigma_1^2 - 2J_1 + \vec{v} \cdot \nabla \sigma_1, \tag{16}$$

$$\rho_2 = \sigma_1 \rho_1 + \vec{v} \cdot \nabla \rho_1, \tag{17}$$

$$\sigma_3 = 2\sigma_1\sigma_2 - 2\dot{J}_1 + \vec{a} \cdot \nabla \sigma_1 + \vec{v} \cdot \nabla \sigma_2, \tag{18}$$

$$\rho_3 = \sigma_2 \rho_1 + \sigma_1 \rho_2 + \vec{a} \cdot \nabla \rho_1 + \vec{v} \cdot \nabla \rho_2, \tag{19}$$

where

$$\dot{J}_1 = \sigma_1 J_1 + \vec{v} \cdot \nabla J_1.$$

The expression (16) is called the stationary flow equation, and (17) is called the stationary vortex equation [3].

Refrences

1. Andronov A. A., Vitt A.A., Khaikin S. E. *Theory of Oscillators.* Moscow, 1966.
2. Jackson. J. D. *Classical Electrodynamics.* USA, 1998.
3. Milne-Thomson L. M., *Theoretical Hydrodynamics.* England, 1968.