CURVILINEAR PARALLELOGRAM IDENTITY AND MEAN-VALUE PROPERTY FOR A SEMILINEAR HYPERBOLIC EQUATION OF SECOND ORDER

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In the domain $\Omega \subseteq \mathbb{R}^2$ of two independent variables $\mathbf{x} = (x_1, x_2) \in \Omega \subseteq \mathbb{R}$ consider the semilinear hyperbolic equation of second-order

$$Au(x_1, x_2) = f(x_1, x_2, u(x_1, x_2), \partial_{x_1} u(x_1, x_2), \partial_{x_2} u(x_1, x_2)), \tag{1}$$

where the operator A is defined as

$$Au(x_1, x_2) := a(x_1, x_2)\partial^2_{x_1} u(x_1, x_2) + 2b(x_1, x_2)\partial_{x_1}\partial_{x_2} u(x_1, x_2) + c(x_1, x_2)\partial^2_{x_2} u(x_1, x_2),$$

and is hyperbolic (this means $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ for any $x \in \Omega$).

Eq. (1) has two families of characteristics: $\gamma_1(x_1, x_2)$ and $\gamma_2(x_1, x_2)$, which are the first integrals of the ordinary differential equation [1]

$$a(\mathbf{x})(\mathrm{dx}_2)^2 - 2\mathrm{b}(\mathbf{x})\mathrm{dx}_1\mathrm{dx}_2 + \mathrm{c}(\mathbf{x})(\mathrm{dx}_1)^2 = 0, \tag{2}$$

and solutions of the equation of characteristics [1]

$$a\left(\frac{\partial \gamma_i}{\partial x_1}\right)^2 + 2b\frac{\partial \gamma_i}{\partial x_1}\frac{\partial \gamma_i}{\partial x_2} + c\left(\frac{\partial \gamma_i}{\partial x_2}\right)^2 = 0, \quad i = 1, 2.$$

It is known [1] that Eq. (2), generally speaking, can be decomposed into two equations. Therefore, we can assume that γ_1 and γ_2 are the first integrals of different differential equations and they are functionally independent since the Jacobian $\left|\frac{\partial(\gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|$ is nonzero [1].

Definition 1. Curvilinear characteristic parallelogram of Eq. (1) is a set $\Pi = \{\mathbf{x} \,|\, \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$, where l_1 , l_2 , r_1 , r_2 are some real numbers and γ_i , $i = 1, 2$ are two different functionally independent characteristics.

Definition 1 is correct. It is known [2] that any other the first integral of Eq. (2) has the form $q \circ \gamma_1$, where q is some continuously differentiable function. If $\gamma_1(\mathbf{x}) \in [l_1, l_2]$, then, due to the continuity of q , $q(\gamma_1(\mathbf{x})) \in q([l_1, l_2]) = [\widetilde{l_1}, \widetilde{l_2}]$. So the curvilinear characteristic parallelogram does not depend on considered characteristics.

Definition 2. Vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \,|\, \gamma_1(\mathbf{x}) \in \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are points $\mathbf{x}$ such that $\gamma_1(x) = l_i \wedge \gamma_2(x) = r_j$, $(i, j) \in \{1, 2\} \times \times \{1, 2\}$.

Definition 2 is correct. We should show that $q \circ \gamma_1$, where q is some continuously differentiable function, maps $[l_1, l_2]$ into $[\widetilde{l_1}, \widetilde{l_2}]$ and $\partial([l_1, l_2])$ into $\partial([\widetilde{l_1}, \widetilde{l_2}])$. Obviously, if the function q is increasing or decreasing these mappings must be true. But if the the function q does not satisfy these conditions, then there exists at least one point $l_0 \in (l_1, l_2)$ such that $q'(l_0) = 0$. Due to the continuity of q , there exists a point $\mathbf{x} \in \Pi$ such that $\gamma_1(\mathbf{x}) = l_0 \in (l_1, l_2)$ This implies

$$\left|\frac{\partial(q \circ \gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|(\mathbf{x}) = \begin{vmatrix} q'(\gamma_1(\mathbf{x}))\partial_{x_1}\gamma_1(\mathbf{x}) & q'(\gamma_1(\mathbf{x}))\partial_{x_2}\gamma_1(\mathbf{x}) \\ \partial_{x_1}\gamma_2(\mathbf{x}) & \partial_{x_2}\gamma_2(\mathbf{x}) \end{vmatrix} = 0 \text{ when } \gamma_1(\mathbf{x}) = l_0.$$

But we consider only characteristics with nonzero Jacobian.

Definition 3. Opposite vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are its vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\gamma_1(\mathbf{x}_1) \neq = \gamma_1(\mathbf{x}_2)$ and $\gamma_2(\mathbf{x}_1) \neq \gamma_2(\mathbf{x}_2)$.

Point transformation of variables of the form $y_1 = \gamma_1(x_1, x_2)$, $y_1 = \gamma_2(x_1, x_2)$ is invertible [3], i.e. there is an inverse change of variables $x_1 = \gamma_1^{-1}(y_1, y_2)$, $x_2 = \gamma_2^{-1}(y_1, y_2)$.

Let us introduce the notation

$$\begin{aligned}
&\beta = 2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right),\\
&K(\mathbf{z}, p, q, r) = f(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}), p, q, r) -\\
&- A\gamma_1(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))(q\partial_{y_1}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_1}\gamma_2^{-1}(\mathbf{z})) -\\
&- A\gamma_2(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))(q\partial_{y_2}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_2}\gamma_2^{-1}(\mathbf{z})),\\
&\widetilde{K}(\mathbf{z}, p, q, r) = (\beta(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})))^{-1}K(\mathbf{z}, p, q, r)
\end{aligned}$$

Theorem 1. *Let the conditions* $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, $f \in C^1(\Omega \times \times \mathbb{R}^3)$, *and* $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ *be satisfied, and let the function* u *belong to the class* $C^2(\Omega)$ *and be a solution of Eq. (1). Then for any curvilinear characteristic parallelogram* $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\} \subseteq \Omega$ *with vertices* $A(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$, $B(\gamma_1^{-1}(l_1, r_2), \gamma_2^{-1}(l_1, r_2))$, $C(\gamma_1^{-1}(l_2, r_2), \gamma_2^{-1}(l_2, r_2))$, $D(\gamma_1^{-1}(l_2, r_1), \gamma_2^{-1}(l_2, r_1))$, *the equality*

$$\begin{aligned}
&u(A) - u(B) + u(C) - u(D) =\\
&= \int_{l_1}^{l_2} dz_1 \int_{r_1}^{r_2} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2
\end{aligned} \tag{3}$$

holds.

Theorem 2. *Let the conditions* $u \in C^2(\Omega)$, $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, $f \in \in C^1(\Omega \times \mathbb{R}^3)$, *and* $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ *be satisfied. If for any curvilinear characteristic parallelogram* $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\} \subseteq \Omega$ *with vertices* $A(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$, $B(\gamma_1^{-1}(l_1, r_2), \gamma_2^{-1}(l_1, r_2))$, $C(\gamma_1^{-1}(l_2, r_2), \gamma_2^{-1}(l_2, r_2))$, $D(\gamma_1^{-1}(l_2, r_1), \gamma_2^{-1}(l_2, r_1))$, *where* γ_i, $i = 1, 2$ *are solutions of Eqs. (2) and* γ_i^{-1} *are defined as before, the equality (3) is satisfied, then the function* u *is a solution of Eq. (1).*

The talk is based on a recent paper [4].

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REDUCTION OF SOME EVOLUTIONARY EQUATIONS BY MEANS OF SYMMETRIES

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Reversible transformations for partial differential equations in the case of two independent variables t, x and one dependent variable $u = u(t, x)$, have the form [1-4]

$$\widetilde{t} = \varphi(t, x, u, \varepsilon), \quad \widetilde{x} = \psi(t, x, u, \varepsilon), \quad \widetilde{u} = \chi(t, x, u, \varepsilon), \tag{1}$$

where

$$\varepsilon \in \mathbb{R},$$

with

$$\varphi(t, x, u, 0) = t, \quad \psi(t, x, u, 0) = x, \quad \chi(t, x, u, 0) = u. \tag{2}$$

The construction of the symmetry group is equivalent to the definition of its infinitesimal transformations

$$\widetilde{t} = t + \varepsilon\tau(t, x, u) + ..., \quad \widetilde{x} = x + \varepsilon\xi(t, x, u) + ..., \quad \widetilde{u} = u + \varepsilon\eta(t, x, u) + ..., \tag{3}$$

where

$$\tau(t, x, u) = \frac{\partial\varphi(t, x, u, 0)}{\partial\varepsilon}, \quad \xi(t, x, u) = \frac{\partial\psi(t, x, u, 0)}{\partial\varepsilon}, \quad \eta(t, x, u) = \frac{\partial\chi(t, x, u, 0)}{\partial\varepsilon}. \tag{4}$$

Consider the k-th order evolutionary equations:

$$u_t + A_0(t, x, u) + \sum_{i=1}^{k} A_i(t, x, u) u_{ix} = 0, \tag{5}$$

where

$$u_{ix} = \frac{\partial^i u}{\partial x^i}, \quad k = 2, 3, 4.$$

Theorem. *For equation (5), the symmetry transformations (1) have the form*

$$\widetilde{t} = \varphi(t, \varepsilon), \quad \widetilde{x} = \psi(t, x, \varepsilon), \quad \widetilde{u} = \chi(t, x, u, \varepsilon), \tag{6}$$