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SHORT COMMUNICATIONS

To the Analysis of Nonlocal Problems of the Theory of Nonlinear Differential Systems

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Abstract—We propose a constructive approach to proving the existence of solutions of operator equations in a Banach space. This approach is applied to the construction and analysis of solutions of matrix Riccati equations bounded on the half-line and not containing a linear term.

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This work is a continuation of studies in [1–5]. It develops an operator approach to the study of solutions of nonlinear matrix differential equations bounded on the half-line.

Consider a nonlinear operator equation

$$x = A(x), \tag{1}$$

where the operator A acts from a Banach space X into a Banach space X and is defined in an open ball $S_\delta(0) \subset X$; 0 is the zero element of the space X, $A(0) \neq 0$, and $\delta > 0$.

Let the operator $A(x)$ be continuously differentiable in $S_\delta(0)$, and let the following estimate be satisfied:

$$\|A'(x)\| \le a\|x\| \quad \text{for each} \quad x \in S_\delta(0), \tag{2}$$

where $a(s) \ge 0$ is a function of the class $C[0,\delta)$ nondecreasing on the interval $[0,\delta)$.

We adopt the notation

$$b = \|A(0)\|, \quad \varphi(\rho, b) = \int\limits_0^\rho \big(a(s) - 1\big)\, ds + b,$$

where $\rho \in [0,\delta)$.

Theorem. *Let the equation $a(\rho) - 1 = 0$ have a solution ρ^* in the interval $(0,\delta)$, and let the inequality*

$$\varphi(\rho^*, b) < 0 \tag{3}$$

hold true. Then Eq. (1) has a unique solution x^* in the closed ball $\bar{S}_\rho(0)$ for each

$$\rho \in \big[\rho_1(b), \rho^*\big), \tag{4}$$

where $\rho_1(b)$ is solution to the equation

$$\varphi(\rho, b) = 0. \tag{5}$$

The solution x^* can be obtained as the limit of a sequence (x_k) whose terms are determined by the classical successive approximation method.

Proof. To prove the theorem, we apply a constructive method (see [1–5]) based on the Caccioppoli–Banach contraction mapping principle [6, p. 605].

Based on the Lagrange formula (see, e.g., [7, p. 375]), we have the equality

$$A(x) = A(0) + \int\limits_0^1 A'(\mu x)\, d\mu x. \tag{6}$$

Then Eq. (1) takes the form

$$x = \int_0^1 A'(\mu x)\, d\mu x + A(0).$$

Take an arbitrary element $x \in \bar{S}_\rho(0)$. Then, using (2), we find estimates in the norm in (6) on the ball $\bar{S}_\rho(0)$,

$$\begin{aligned}\|A(x)\| &\le \left\| \int_0^1 A'(\mu x)\, d\mu \right\| \|x\| + \|A(0)\| \le \int_0^1 \big\|A'(\mu x)\big\|\, d\mu \|x\| + b \\ &\le \int_0^1 a\mu\|x\|\, d\mu\|x\| + b \le \int_0^1 a\mu\rho\, d\mu\rho + b = \int_0^\rho a(s)\, ds + b.\end{aligned}$$

Consider Eq. (5). It follows from the relation $\varphi(0,b) > 0$ and condition (3) that it has a solution in the interval $(0,\rho^*)$.

In the interval $[0,\rho^*)$, we have the inequality

$$\frac{d\varphi(\rho,b)}{d\rho} \equiv a(\rho) - 1 < 0.$$

Since the function $\varphi(\rho,b)$ is continuous on the interval $[0,\rho^*]$, it follows that Eq. (5) has a unique solution $\rho_1 = \rho_1(b)$ in the interval $(0,\rho^*)$. It can be seen from the relation

$$\varphi(\rho,b) = \int_{\rho_1}^\rho \big(a(s) - 1\big)\, ds$$

that for the values of ρ belonging to domain (4) one has the inequalities

$$\int_0^\rho \big(a(s) - 1\big)\, ds + b \le \rho, \tag{7}$$

$$a(\rho) < 1, \tag{8}$$

which will be used to analyze the contraction property of the operator A. From Eq. (1) for any $x \in \bar{S}_\rho(0)$ and $y \in \bar{S}_\rho(0)$, we have the relation

$$A(x) - A(y) = \int_0^1 A'\big(y + \mu(x-y)\big)\, d\mu(x-y).$$

After performing estimates in the norm, we successively obtain

$$\begin{aligned}\big\|A(x) - A(y)\big\| &\le \int_0^1 \Big\|A'\big(y + \mu(x-y)\big)\Big\|\, d\mu\|x-y\| \\ &\le \int_0^1 a\big\|(1-\mu)y + \mu x\big\|\, d\mu\|x-y\| \\ &\le \int_0^1 a(1-\mu)\|y\| + \mu\|x\|\, d\mu\|x-y\| \le a(\rho)\|x-y\|.\end{aligned}$$

Based on inequality (8), we conclude that the operator A is a contraction on the ball $\bar{S}_\rho(0)$. To complete the proof of this part of the theorem, it suffices to refer to the Theorem in [6, p. 605].

To construct solutions of Eq. (1), we use the well-known algorithm (see, e.g., [6, p. 605])

$$x_{n+1} = A(x_n) \quad (n = 0, 1, 2, \ldots), \tag{9}$$

where x_0 is an arbitrary element in $\bar{S}_\rho(0)$. By virtue of (7), it can readily be shown that all terms of the sequence (x_k) belong to $\bar{S}_\rho(0)$.

Next, we obtain an estimate characterizing the rate of convergence of the sequence (x_k) to the solution x^*. From (9), we have

$$x_{k+1} - x_k = A(x_k) - A(x_{k-1}) \quad (k = 1, 2, \ldots). \tag{10}$$

Based on the Lagrange formula, relation (10) can be written as follows:

$$x_{k+1} - x_k = \int\limits_0^1 A'\big(x_{k-1} + \mu(x_k - x_{k-1})\big)\, d\mu (x_k - x_{k-1}). \tag{11}$$

In (11), we perform estimates with respect to the norm

$$\begin{aligned} \|x_{k+1} - x_k\| &\le \int\limits_0^1 \Big\|A'(x_{k-1} + \mu\big(x_k - x_{k-1})\big)\Big\|\, d\mu \|x_k - x_{k-1}\| \\ &\le \int\limits_0^1 a\big\|(1-\mu)x_{k-1} + \mu x_k\big\|\, d\mu \|x_k - x_{k-1}\| \\ &\le \int\limits_0^1 a\big((1-\mu)\|x_{k-1}\| + \mu\|x_k\|\big)\, d\mu \|x_k - x_{k-1}\| \le a(\rho)\|x_k - x_{k-1}\|. \end{aligned}$$

Thus, we have obtained the estimate

$$\|x_{k+1} - x_k\| \le a(\rho)\|x_k - x_{k-1}\| \quad (k = 1, 2, \ldots). \tag{12}$$

Using (12), on the basis of (7), (8) one can readily prove that the sequence (x_k) converges to an element $x^* \in \bar{S}_\rho(0)$, with the following estimate holding true:

$$\|x^* - x_k\| \le \frac{a^k}{1-a}\|x_1 - x_0\| \quad (k = 1, 2, \ldots). \tag{13}$$

Remark 1. Obviously, in the proved theorem, instead of the quantity $b = \|A(0)\|$, we can take an estimate for $\|A(0)\|$.

Remark 2. The equation $\alpha(\rho) - 1 = 0$ has a unique solution in the interval $(0, \delta)$ under the condition $\alpha(s_0) < 1 < \alpha(s_1)$, where $0 \le s_0 < s_1 < \delta$.

Using the Theorem, we study the question of the existence of solutions of the matrix Riccati equation bounded on the half-line $\mathbb{R}_+ \equiv [0, \infty)$ (see [8; 9, p. 165; 10, p. 158], etc.),

$$\frac{dY}{dt} = YP(t)Y + Q(t), \tag{14}$$

where $P(t)$ and $Q(t)$ are continuous $n \times n$ matrices bounded in $\mathbb{R}_+$ and satisfying the conditions

$$\tilde{p} \equiv \int\limits_0^\infty \big\|P(\tau)\big\|\, d\tau < \infty, \quad \tilde{q} \equiv \sup_{t \ge 0} \big\|\tilde{Q}(t)\big\| < \infty;$$

here $\tilde{Q}(t) = \int_0^t Q(\tau)\, d\tau$.

Set

$$\|Y\|_C \equiv \sup_{t\ge 0} \big\|Y(t)\big\|,$$

where $C = \mathfrak{B}(n)$ is a finite-dimensional Banach algebra of matrix functions continuous and bounded on the half-line and $\|\cdot\|$ is a certain matrix norm, for example, any of the norms given in [11, p. 21].

For Eq. (14), we will study the Cauchy problem with the condition

$$Y(0) = \Lambda. \tag{15}$$

Instead of problem (14), (15) we consider the equivalent integral equation

$$Y(t) = \Lambda + \int_0^t Y(\tau)P(\tau)Y(\tau)\,d\tau + \tilde{Q}(t). \tag{16}$$

We investigate the solvability of this equation in $\mathfrak{B}(n)$; the convergence of a sequence means uniform convergence on the half-line $\mathbb{R}_+$.

For every $n\times n$ matrix $X(t)$ belonging to the ball $\|X\|_C \le \rho$, we have

$$\left\|\int_0^t X(\tau)P(\tau)X(\tau)\,d\tau + \Lambda + \tilde{Q}(t)\right\|_C \le \tilde{p}\rho^2 + \varepsilon + \tilde{q},$$

where $\varepsilon = \|\Lambda\|$.

In a similar manner, we obtain the estimate

$$\left\|\int_0^t \big(X(\tau)P(\tau)X(\tau) - Y(\tau)P(\tau)Y(\tau)\big)\,d\tau\right\|_C \le 2\tilde{p}\rho\|X-Y\|_C,$$

where $\|X\|_C \le \rho$ and $\|Y\|_C \le \rho$.

As applied to Eq. (16), we have

$$a(\rho) = 2\tilde{p}\rho,$$
$$\varphi(\rho,b) = \tilde{p}\rho^2 - \rho + b,$$

where $b = \varepsilon + \tilde{q}$.

Since $\rho^* = 1/(2\tilde{p})$, condition (3) takes the form

$$\varphi(\rho^*, b) = b - \frac{1}{4\tilde{p}} < 0.$$

It follows from the theorem proved that under the condition

$$\tilde{q} - \frac{1}{4\tilde{p}} < 0$$

the problem of solutions of Eq. (14) bounded on the half-line is uniquely solvable for initial values belonging to the domain

$$\|\Lambda\| < \frac{1}{4\tilde{p}} - \tilde{q}.$$

In this case,

$$\rho_1(\varepsilon) \le \rho < \frac{1}{4\tilde{p}};$$

here

$$\rho_1(\varepsilon) = \frac{1 - \sqrt{1 - 4\tilde{p}(\varepsilon + \tilde{q})}}{2\tilde{p}} > \frac{1 - \sqrt{1 - 4\tilde{p}\tilde{q}}}{2\tilde{p}} = \rho_1(0).$$

To construct a solution of Eq. (16), one can use the algorithm (9) together with the estimate (13).

Remark 3. The above theorem is stated and proved in terms of the functions $a(s)$ and $\varphi(\rho, b)$; this is its constructiveness, which is illustrated by the example of the Riccati equation (and earlier in [1–5]), for which coefficient sufficient conditions for the existence of solutions bounded on the half-line $\mathbb{R}_+$ are obtained.

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