A TOPOLOGICAL INDEX OF FULLER TYPE FOR PERIODIC ORBITS

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We want to give a short overview on the basic concepts of the topological index theory for periodic orbits. The problem of establishing the existence of periodic solution for a differential equation $x'(t)=f(x(t))$ has been studied by many authors over the years.

There are many methods for solving this problem; for example we can look for periodic orbits by the use of fixed point theory, the Conley index, the Fuller index, etc. Historically, the first definition of an index counting the periodic orbits of a smooth vector field was given by F. B. Fuller in 1967 (see [1]). Fuller's article contains important and relevant ideas for the approach to this problem. Briefly, nowadays there are two approaches to defining the Fuller index. The first one, analytical, was considered e.g. by Chow and Mallet-Paret (see [2]); it required appropriate smoothness of the maps involved and relied on a complicated bifurcation argument.

A different approach has been presented by Potter (see [3,4]). His construction is similar to the construction of a topological degree for A–proper maps in Banach spaces and he was in a position to apply his index to some differential equations defined in infinite-dimensional Banach spaces; unfortunately, this index was not defined to be a rational number – it was a sequence of rational numbers. Furthermore, an approach due to Fenske is similar to that presented in the original paper of Fuller (see [5]). He adapted Fuller's construction to some classes of flows defined on Banach spaces. More exactly, he extended the Fuller index to a class of flows generated by functional differential equations.

There is yet another approach to the Fuller index involving algebraic topology methods – homology or cohomology theory. In his paper, Fuller applied the cohomology of differential forms (see [1]). He assumed that the right–hand side of a differential equation under study is defined on a smooth and finite–dimensional manifold. Franzosa defined a new homological index of the Fuller type (see [6]). It has all the expected properties: existence, additivity, homotopy invariance and normalization, and provides more information about periodic orbits than the original Fuller invariant; however Franzosa defined his index only for flows generated by differential equations defined on smooth finite–dimensional manifolds. Fenske tried to avoid this assumption and constructed the Fuller index for flows defined on an arbitrary ANR; moreover he avoided the smoothness assumptions (see [7–9]). Unfortunately, his construction is quite complicated and it is not clear how to apply it to differential equations.

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A totally different homological approach to the Fuller index was presented by Srzednicki (see [10–12]). He used the fixed point transfer approach of fiber-preserving maps due to Dold (see [13, 14]). Finally, it should be noted that Crabb and Potter (see [15]) proposed a construction of the Fuller index by the use of the fiberwise stable homotopy theory. However, from our viewpoint, the results of Srzednicki are the most important.

The purpose of this talk is to present an invariant responsible for the existence of periodic orbits of differential equations of the form $x'(t)=f(x(t))$, where $f : Rn \to Rn$ is only continuous, i.e., equations without the uniqueness of solutions property. In this case the flow generated by the problem is multivalued and its dynamics may be fairly complicated. It requires special attention, since – as it seems – it cannot be adequately studied within a framework provided by any of the above mentioned papers.

This is a joint work with Wojciech Kryszewski. The presented results are contained in the following papers:

– W. Kryszewski, R. Skiba, A cohomological index of Fuller type for multivalued dynamical systems, Nonlinear Analysis, 75, 684–716 (2012);

– R. Skiba, A cohomological index of Fuller type for parameterized set-valued maps in normed spaces, Cent. Eur. J. Math. 12, no. 8, 1164–1197 (2014);

– R. Skiba, The transversal degree for Fredholm maps of positive index I: A cohomological approach (in preperation);

– R. Skiba, The transversal degree for Fredholm maps of positive index II: Applications (in preperation).

Finally, I recommend the following articles for the further deeper study.

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