ON SOLUTION OF A CASE OF $\mathbb{R}$-LINEAR CONJUGATION PROBLEM BY THE METHOD OF MATRIX-FUNCTIONS FACTORIZATION

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A generalization of G.P. Chebotarev approach to the factorization of triangular matrix functions [1] was recently proposed by first two authors [6]. Later, it was discovered that this generalized method can be applied to the factorization of more general matrix-functions. In our report we describe how this method can be used for vector-matrix form of the $\mathbb{R}$-linear conjugation problem. Applications of the $\mathbb{R}$-linear conjugation problem to the study of elasticity problem [5] as well as to answering the questions from mechanics of composite materials [3, 4] are well known.

We consider a case of the $\mathbb{R}$-linear conjugation problem (or Markushevich problem) on the unit circle

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + f(t), \quad t \in \mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}, \tag{1}$$

where $\varphi^+(t)$, $\varphi^-(t)$ are boundary values of unknown functions, analytic inside (i.e. in $D^+ := \mathbb{D}$) and outside (i.e. in $D^- := \mathbb{C}\setminus\overline{\mathbb{D}}$) of the unit disc $\mathbb{D}$. Note that we consider here only the elliptic case $(|a(t)| > |b(t)|)$ of the problem (1), see [2]. Let $\varkappa = \mathrm{ind}_{\mathbb{T}} a(t)$ be the Cauchy index of the coefficient $a(t)$. Then the problem (1) can be equivalently reduced to the following boundary value problem

$$\psi^+(t) = t^{\varkappa}\psi^-(t) + p(t)\overline{\psi^-(t)} + h(t), \quad t \in \mathbb{T}, \tag{2}$$

where $p(t)$ is a boundary function analytically extendible into D^-. As it was shown in [2] such problem on the unit circle is equivalent to the vector-matrix $\mathbb{C}$-linear conjugation problem

$$\Psi^+(t) = \begin{pmatrix} t^{\varkappa} & 0 \\ 0 & t^{\varkappa} \end{pmatrix} \begin{pmatrix} 1 - p(t)\overline{p(t)} & p(t) \\ -\overline{p(t)} & 1 \end{pmatrix} \Psi^-(t) + H(t), \quad t \in \mathbb{T}, \tag{3}$$

We study the problem (2) (or (4)) under the following additional assumption. Let us additionally assume that the function $p(t)$ is a finite segment of the Fourier series

$$p(t) = \sum_{n=k}^{m} \frac{c_n}{t^n} =: \frac{C_{m-k}(t)}{t^m}. \tag{4}$$

Solution of the vector-matrix boundary value problem (4) is equivalent (see [5]) to the factorization of the matrix

$$A(t) = \begin{pmatrix} 1 - p(t)\overline{p(t)} & p(t) \\ -\overline{p(t)} & 1 \end{pmatrix}, \tag{5}$$

i.e. its representation in the form

$$A(t) = A^+(t)\Lambda(t)A^-(t),$$

where $A^{\pm}(t)$ admit together with their inverse $(A^{\pm}(t))^{-1}$ analytic extension into $D^{\pm}$, respectively, and $\Lambda(t) = \operatorname{diag}\{t^{\varkappa_1}, t^{\varkappa_2}\}$ is a diagonal matrix. The integer numbers $\varkappa_1$, $\varkappa_2$ in this representation are called partial indices of the matrix $A(t)$ and $A^{+}(t)$, $A^{-}(t)$ are plus- and minus-factors, respectively.

It is known (see, e.g. [5]) that factorization of this type is equivalent to the construction of the canonical matrix $X^{\pm}(z)$. It means that these matrices are analytic in the corresponding domains, satisfy the following boundary condition

$$X^{+}(t) = A(t)X^{-}(t), \quad t \in \mathbb{T}, \tag{6}$$

and $X^{-}(z)$ has a normal form at infinity (i.e. the sum of the orders of its columns is equal to $\operatorname{ind}_{\mathbb{T}} \det A(t)$.)

To construct canonical matrix we start with the pair of matrices

$$X_0^{+}(t) = E_2, \quad X_0^{-}(t) = \begin{pmatrix} 1 & -p(t) \\ \overline{p(t)} & 1 - p(t)\overline{p(t)} \end{pmatrix},$$

which satisfy the boundary condition (6). It follows from the above assumption (4) that $X_0^{-}(z)$ does not have the normal form at infinity. In order to achieve normality we change the columns of $X_0^{-}(z)$ by multiplying from the right both sides of (6) on the triangular polynomial matrix functions

$$P(t) = \begin{pmatrix} 1 & 0 \\ Q(t) & 1 \end{pmatrix} \quad \text{or} \quad P(t) = \begin{pmatrix} 1 & Q(t) \\ 0 & 1 \end{pmatrix},$$

where the polynomials $Q(t)$ are found at the expansion of the ratio $1/p(t)$ into continuous fraction. In our case this expansion is finite and after finite number of steps we obtain from $X_0^{-}(z)$ a sequence of matrices $X_0^{-}(z)$, $X_1^{-}(z)$, $\ldots$, $X_{\nu+1}^{-}(z)$ with the last one represented in one of the following forms

$$X_{\nu+1}^{-}(t) = \begin{pmatrix} C_{r_\nu}(t)/t^m & 0 \\ F_\nu(t) & F_{\nu+1}(t) \end{pmatrix} \quad \text{or} \quad X_{\nu+1}^{-}(t) = \begin{pmatrix} 0 & -C_{r_\nu}(t)/t^m \\ F_{\nu+1}(t) & F_\nu(t) \end{pmatrix},$$

where $C_{r_\nu}(t)$ are certain polynomials of order r_ν, and F_1, F_2, $\ldots$, F_ν, $F_{\nu+1}$ are rational functions having the order at infinity respectively d_1, d_2, $\ldots$, d_ν, $d_{\nu+1}$. Applying to the these triangular matrices a generalization of the Chebotarev method [6] we obtain the following result.

Theorem. *Let for certain* $k, 1 \leqslant k \leqslant \nu$, *the numbers* d_k *be satisfied inequalities*

$$d_1 < 0, \quad \ldots, \quad d_{k-1} < 0, \quad d_k \geqslant 0, \tag{7}$$

then

(i) if $d_k = 0$, *then partial indices* $\varkappa_1$, $\varkappa_2$ *of the matrix* $A(t)$ *are vanishing, i.e.* $\varkappa_1 = \varkappa_2 = 0$;

(ii) if $d_k > 0$, *then* $\varkappa_{1,2} = \pm\min\{m - r_k, d_k\}$.

Remark. *In the remaining case* $d_1 < 0$, $\ldots$, $d_\nu < 0$, $d_{\nu+1} < 0$ *we can only conclude that partial indices* $\varkappa_1$, $\varkappa_2$ *are finite and belong to the interval* $[k - m + 1, m - k - 1]$.

The obtained results are illustrated by the concrete examples when $p(t)$ contains only two or three terms.

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