TWO-POINT BOUNDARY VALUE PROBLEMS FOR ESSENTIALLY SINGULAR SECOND ORDER DIFFERENTIAL EQUATIONS

I. Kiguradze

On a finite open interval $]a,b[$, we consider the differential equation

$$u'' = f(t,u) \tag{1}$$

with the boundary conditions

$$u(a+) = 0, \quad u(b-) = 0, \tag{2}$$

or

$$u(a+) = 0, \quad u'(b-) = 0, \tag{3}$$

where $f :]a,b[\times\mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, $u(a+)$ is a right limit of the function u at the point a, while $u(b-)$ and $u'(b-)$ are left limits of the functions u and u' at the point b.

A method for the investigation of problem (1), (2) (of problem (1), (4)) is elaborated in the case where the function f has singularities of arbitrary order at the points a and b (a singularity of arbitrary order at the point a) with respect to the time variable.

Based on this method, in particular, the following theorem is proved.

Theorem. *Let in the domain* $]a,b[\times\mathbb{R}$ *the condition*

$$(f(t,x) - f(t,y))\,\mathrm{sgn}\,(x-y) \geqslant p(t)|x-y|$$

hold, where $p :]a,b[\to \mathbb{R}$ *is a Lebesgue integrable on every closed interval contained in* $]a,b[$ *function. Let, moreover,*

$$\int\limits_a^b (t-a)(b-t)[p(t)]_- dt \leqslant b-a,$$

and there exist an absolutely continuous function $\delta :]a,b[\to]0,+\infty[$ *such that* $\delta(a+) = \delta'(a+) = 0$, $\delta(b-) = \delta'(b-) = 0$,

$$\liminf_{t\to a}(\delta^2(t)p(t)) > 0, \quad \liminf_{t\to b}(\delta^2(t)p(t)) > 0, \quad \int\limits_a^b \delta(t)|f(t,0)|\,dt < +\infty.$$

Then problem (1), (2) *has one and only one solition.*

Example. Let

$$f(t,x) = \delta^{-2}(t)p_0(t)\exp(|x|)x + \delta^{-1}(t)q_0(t),$$

where $\delta(t) = \exp(-1/(t-a) - 1/(b-t))$, and p_0, $q_0 : [a,b] \to]0,+\infty[$ are continuous functions. Then for any $\ell > 0$ and $x \in \mathbb{R}$ the condition

$$\int_a^b (t-a)^\ell (b-t)^\ell |f(t,x)|\, dt = +\infty$$

is satisfied, but nevertheless, according to the above theorem, problem (1), (2) has one and only one solution.