LAGRANGE INTERPOLATION FORMULAS FOR OPERATORS WITH RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVES

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We consider the differential operators $F: X \to Y$ containing fractional derivatives of an order α_i $(0 < \alpha_i < 1,\ i = 1, 2, \ldots, m)$ in the form

$$F(x) = f(t, x(t), x^{(\alpha_1)}(t), x^{(\alpha_2)}(t), \ldots, x^{(\alpha_m)}(t)), \tag{1}$$

where

$$x^{(\alpha_i)}(t) \equiv \frac{d^{\alpha_i} x(t)}{dt^{\alpha_i}} = \frac{1}{\Gamma(1-\alpha_i)} \frac{d}{dt} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha_i}}\, d\tau$$

is the Riemann–Liouville fractional derivative [1] of the order α_i, $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, m)$, $m \in \mathbb{N}$; $t \in [a, b] = T \subset \mathbb{R}_+$; $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ is the Gamma function; $X \equiv X(T) = AC[a, b]$ is the space of absolutely continuous functions on a segment $[a, b]$; $x(t) \in X$; the function $y = f(t, u_0, u_1, \ldots, u_m)$ is defined on the rectangle $\Omega = T \times T_0 \times T_1 \times \ldots \times T_m$, T_i are sets of the number line $(i = 0, 1, \ldots, m)$, and Y is the domain of values of the operator $F(x)$.

Let $l_{n,k}(x)$ be the Lagrange fundamental polynomials of the n-degree with respect to arbitrary Chebyshev systems $\{\phi_k(x)\}_{k=0}^n$ of functions relative to nodes $x_0, x_1, \ldots, x_n$, i.e. $l_{n,k}(x_j) = \delta_{kj}$ is the Kronecker symbol $(k, j = 0, 1, \ldots, n)$; $\sigma_n(x) = \sum_{k=0}^n l_{n,k}(x)$ be a constant or a variable value.

Theorem 1. *The polynomial*

$$L_n(F; x) = f(t, x_0(t), x_0^{(\alpha_1)}(t), x_0^{(\alpha_2)}(t), \ldots, x_0^{(\alpha_m)}(t)) +$$
$$+ \sum_{k=1}^{n} \int_0^1 \sum_{\nu=0}^{m} \frac{\partial}{\partial \upsilon_k^{(\alpha_\nu)}} f\left(t, \upsilon_k(t,\tau), \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \upsilon_k(t,\tau), \ldots, \frac{\partial^{\alpha_m}}{\partial t^{\alpha_m}} \upsilon_k(t,\tau)\right) \times$$
$$\times \frac{\partial^{\alpha_\nu}}{\partial t^{\alpha_\nu}} \left\{ \frac{l_{n,k}(x(t))}{\sigma_n(x(t))} (x_k(t) - x_0(t)) \right\} d\tau, \tag{2}$$

where $\alpha_0 \equiv 0$, $\upsilon_k \equiv \upsilon_k(t,\tau) = x_0(t) + \tau(x_k(t) - x_0(t))$, $\upsilon_k^{(\alpha_\nu)} = \partial^{\alpha_\nu} \upsilon_k(t,\tau)/\partial t^{\alpha_\nu}$ $(k = 1, 2, \ldots, n)$, *is the interpolation operator for the operator* $F(x)$ *of the form (1) given on the set* $X(T)$ *relative to the functional nodes* $x_0, x_1, \ldots, x_n \in X$: $L_n(F; x_k) = F(x_k)$ $(k = 0, 1, \ldots, n)$.

Theorem 2. *The interpolation error* $r_n(x) = F(x) - L_n(F; x)$, *where* $F(x)$ *is the operator of the form (1), and* $L_n(F; x)$ *is the polynomial (2), can be represented as*

$$r_n(x) = \sum_{k=1}^{n+1} \int_0^1 \sum_{\nu=0}^{m} \frac{\partial}{\partial \upsilon_k^{(\alpha_\nu)}} f\left(t, \upsilon_k(t,\tau), \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} \upsilon_k(t,\tau), \ldots, \frac{\partial^{\alpha_m}}{\partial t^{\alpha_m}} \upsilon_k(t,\tau)\right) \times$$
$$\times \frac{\partial^{\alpha_\nu}}{\partial t^{\alpha_\nu}} \left\{ \left(\frac{l_{n+1,k}(x(t))}{\sigma_{n+1}(x(t))} - \frac{l_{n,k}(x(t))}{\sigma_n(x(t))} \right) (x_k(t) - x_0(t)) \right\} d\tau,$$

where $x_{n+1} \equiv x$, $l_{n,n+1}(x) \equiv 0$.

The presented results are based on several Lagrange interpolation polynomials obtained earlier by the authors for general operators in functional spaces [2] and may be used in theoretical and applied research as a basis for constructing analytical and numerical methods for solving equations that contain fractional derivatives of Riemann–Liouville.

References

1. Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and applications of fractional differential Equations.* Elsevier Science, Amsterdam, 2006.

2. Yanovich L. A., Ignatenko M. V. *Operator interpolation formulas based on interpolation polynomials for numerical functions* // Proc. Mathematics Institute of NAS of Belarus. 2002. № 11. P. 157–167 (in Russian).