QUANTUM PARTICLE WITH INTRINSIC STRUCTURE IN PRESENCE OF MAGNETIC FIELD ON THE BACKGROUND OF SPHERICAL RIEMANN SPACE

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Generalized Schrödinger equation for spin zero particle with intrinsic structure by Darwin–Cox is studied in presence of magnetic field on the background of 3-dimensional spherical Riemann space. Separation of the variables is performed. An equation describing motion along the axis z is studied. The form of the effective potential curve indicates that we have a quantum-mechanical problem with the complicated box type potential. Frobenius solutions of the equation are constructed, convergence of the relevant series is examined by Poincaré–Perron method. These series are convergent in the all physical domain of the variable $z \in [-\pi/2, +\pi/2]$. Visualization of constructed solutions is performed. Because the spherical space is compact, we assume existence of discrete series of energy levels, however any exact quantization rule is not known. Approximate method for producing the discrete spectrum for energy is developed, it is based of the use of polynomials instead of power series involved in exact Frobenius solutions.

Cox's electromagnetic structure [1] may be related to the known Darwin interaction term in non-relativistic Schrödinger equation, this additional interaction is related to non-point-like distribution of the electric charge in the finite volume inside the particle see, for instance, in the book [2]. In the present paper we will study the problem of Cox particle in magnetic field, but now on the background of spherical Riemann space. In cylindric coordinates of this space we have (non-zero quantity γ is a Cox structure parameter)

$$dS^2 = dt^2 - \cos^2 z(dr^2 + \sin^2 r d\phi^2) - dz^2,$$

$$z \in [-\pi/2, +\pi/2], \quad A_\phi = b(\cos r - 1), \quad F_{r\phi} = b \sin r.$$

After separating the variable in Schrödinger equation we derive equations in r, z-variables:

$$\left(\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \frac{[m + b(\cos r - 1)]^2}{\sin^2 r} + \lambda\right) R(r) = 0,$$

$$\left(\frac{d^2}{dz^2} + \varepsilon + 1 - U(z)\right) f(z) = 0, \quad U(z) = \frac{\beta + \lambda\cos^2 z}{\cos^4 z - \gamma^2}, \quad \beta = b\gamma. \tag{1}$$

Analysis of the radial equation gives a completely discrete energy spectrum (for more detail see in [2]). The goal of the present paper is to study (analytically and numerically) solutions of the equation in z-variable (1). In the variable $x = 1 - Z = \sin^2 z$, we reduce Eq. (1) to the form (for brevity we will use notations $1 - \gamma = s$, $1 + \gamma = t$, and shortening notations for numerical coefficients A, B, C, D)

$$\left(\frac{d^2}{dx^2} + \left(\frac{1/2}{x-1} + \frac{1/2}{x}\right)\frac{d}{dx} - \frac{A}{x-1} - \frac{B}{x} - \frac{C}{x-s} - \frac{D}{x-t}\right) f(Z) = 0.$$

In fact, here we have a box type quantum mechanical problem with a complicated potential, mathematical task reduces to differential equation with six regular singularities. Frobenius solutions are searched in the form $f(x) = x^a(x-1)^b g(x)$, $a = 0, 1/2$, $b = 0, 1/2$. The functions $g_i(x)$ are subject to the equation of one the same general structure

$$g'' + \left(\frac{\alpha_1}{x} + \frac{\beta_1}{x-1}\right) g' + \left(\frac{\alpha}{x} + \frac{\beta}{x-1} - \frac{C}{x-s} - \frac{D}{x-t}\right) g = 0, \tag{2}$$

$$\alpha_1 = \frac{1}{2}(1+4a), \quad \beta_1 = \frac{1}{2}(1+4b), \quad \alpha = \frac{1}{2}(-4ab-2B-a-b), \quad \beta = \frac{1}{2}(4ab-2A+a+b).$$

Solutions are constructed as power series, so we arrive at the 5-term recurrence relations for coefficients. Poincaré–Perrone approach is used to analyze convergence radius of the series. The possible convergence radii are

$$R_{\text{conv}} = 1, \quad |t| = |1+\gamma|, \quad |s| = |1-\gamma|, \quad \infty.$$

Thus, the power series near the point $x = 0$ is guaranteed to converge in a circle of radius $R_{\text{conv}} = 1 - |\gamma|$; this is the most interesting area from the physical standpoint. Perhaps convergence will be extended further, due to non-singular behavior of the solutions near the points $|x| = |1 \pm \gamma|$.

We have no analytical rule to get quantized energy levels. However, it turns out that exact polynomial solutions exist for physical regions of energy parameter ε.

To get some description for physical energy level, we will apply such polynomial approximations for describing solutions of Eq. (2). First we set $a = 0, b = 0$ and take polynomials as exact solutions of Eq. (2). For definiteness, let the values of parameter be $\lambda = 1$, $\beta = 10^{-2}$, $\gamma = 10^{-1}$, then Eq. (2) takes the form

$$\frac{d^2}{dx^2}g + \left(\frac{1}{2x} + \frac{1}{2(x-1)}\right)\frac{d}{dx}g +$$

$$+\left(-\frac{(-\varepsilon-1)/4 + 91/360}{x} - \frac{(\varepsilon+1)/4 + 1/40}{x-1} - \frac{9}{8(x-11/10)} + \frac{101}{72(x-9/10)}\right) g = 0.$$

Let $g(x) = g_{10}(x)$, remembering on the structure of power series, we substitute this 10-degree polynomial into Eq. (2). As a result we obtain a polynomial equation of the third order, with zero coefficient at x^0. Hence we have a system of the three equations each with 11 real roots:

$$\begin{aligned}
x^3:&\quad 1.66, \quad 10.40, \quad 16.27, \quad 34.72, \quad 62.90,\\
&\quad 99.05, \quad 143.06, \quad 195.03, \quad 255.01, \quad 323.001, \quad 399.;\\
x^2:&\quad 1.65, \quad 9.93, \quad 16.13, \quad 34.81, \quad 62.94,\\
&\quad 99.05, \quad 143.04, \quad 195.02, \quad 255.015, \quad 323.001, \quad 399.;\\
x:&\quad 1.63, \quad 9.53, \quad 16.01, \quad 34.86, \quad 62.97,\\
&\quad 99.05, \quad 143.05, \quad 195.01, \quad 255.052, \quad 323.001, \quad 399.
\end{aligned}$$

Now, let $g(x) = g_{15}(x)$, we substitute this 15-order polynomial into Eq. (2), this yields again three equations, which lead to (close to each other) three series of 15 real energy levels:

x^3 : 1.76, 13.86, 20.80, 32.54, 61.93, 98.77, 143.14,
195.18, 255.11, 323.09, 399.04, 483.055, 575.00, 675., 783., 899.;
x^2 : 1.75, 13.63, 19.83, 33.03, 62.11, 98.84, 143.15,
195.163, 255.09, 323.04, 399.01, 483.00, 575.00, 675., 783., 899.;
x : 1.744, 13.42, 19.13, 33.36, 62.24, 98.88, 143.14,
195.14, 255.08, 323.03, 399.01, 483.00, 575., 675., 783., 899.

We have made similar calculation with the use of polynomial P_{20} and P_{25}, the tendency remains the same. We may think that that exactness becomes better when the order of polynomials increases. So, a simple approximate method for getting the discrete spectrum for energy is developed, it is based of the use of polynomials instead of power series involved in exact Frobenius solutions.

References

1. Cox W. *Higher-rank representations for zero-spin field theories* // J. Phys. Math. Gen. 1982. V. 15. P. 627–635.

2. Kisel V. V., Ovsiyuk E. M., Veko O. V., Voynova Y. A., Balan V., Red'kov V. M. *Elementary Particles with Internal Structure in External Fields. V. II. Physical Problems.* New York: Nova Science Publishers Inc., 2018.